Math 171 **Solutions to homework problems** **Spring 2005**

Section 5.4

1 Evaluate the following improper integrals.

(b) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \lim_{c\to-\infty}\lim_{d\to+\infty}\int_c^d \frac{1}{1+x^2}dx = \lim_{c\to-\infty}\lim_{d\to+\infty}(\arctan d - \arctan c)$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

(c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}}dx = \lim_{c\to 0^+}\int_c^{\pi/2}\frac{\cos x}{\sqrt[3]{\sin x}}dx = \lim_{c\to 0^+}\int_{\sin c}^1 \frac{1}{\sqrt[3]{u}}du = \lim_{c\to 0^+}\left.\frac{u^{2/3}}{2/3}\right|_{\sin c}^1$

$= \lim_{c\to 0^+}\left(\frac{3}{2} - \frac{3(\sin c)^{2/3}}{2}\right) = \frac{3}{2}$ (substitution: $u = \sin x$)

2(b) Find all values of $p \in \mathbb{R}$ for which $f(x) = \frac{1}{x^p}$ is improperly integrable on $I = (0,1)$.

if $p \neq 1$*, then* $\int_0^1 \frac{1}{x^p}dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x^p}dx = \lim_{c\to 0^+}\left.\frac{x^{1-p}}{1-p}\right|_c^1 = \lim_{c\to 0^+}\left(\frac{1}{1-p} - \frac{c^{1-p}}{1-p}\right)$.

The limit is finite if and only if $1-p>0$*, i.e.* $p<1$.

If $p=1$*, then* $\int_0^1 \frac{1}{x}dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x}dx = \lim_{c\to 0^+}\ln x|_c^1 = \lim_{c\to 0^+}(\ln 1 - \ln c) = \infty$.

Therefore the function $f(x) = \frac{1}{x^p}$ *is improperly integrable on* $I=(0,1)$ *if and only if* $p<1$.

4(e) Decide whether $f(x) = \frac{1-\cos x}{x^2}$ is improperly integrable on $I = (0,\infty)$.

Yes. $\int_0^\infty \frac{1-\cos x}{x^2}dx = \int_0^1 \frac{1-\cos x}{x^2}dx + \int_1^\infty \frac{1-\cos x}{x^2}dx$. *We will show that each of these two integrals converges.*

On $(0,1]$, $g(x) = x^2+\cos x - 1 > 0$ *since* $g(0)=0$ *and* $g'(x) = 2x - \sin x > 0$ *for* $x \in (0,1]$ *(the latter holds because* $g'(0)=0$ *and* $g''(x) = 2-\sin x > 0$*).*

Therefore $1-\cos x < x^2$*, so* $\frac{1-\cos x}{x^2} < 1$. *Therefore* $\int_0^1 \frac{1-\cos x}{x^2}dx$ *converges.*

On $[1,\infty)$, $f(x) = \frac{1-\cos x}{x^2} \leq \frac{2}{x^2}$ *and* $\int_1^\infty \frac{1}{x^2}dx$ *converges, therefore* $\int_1^\infty \frac{1-\cos x}{x^2}dx$ *converges.*

5 Use the examples provided by Exercise 2b to show that the product of two improperly integrable functions might not be improperly integrable.

$\frac{1}{x^{1/2}}$ *is improperly integrable on* $(0,1)$*, but* $\frac{1}{x^{1/2}}\cdot\frac{1}{x^{1/2}} = \frac{1}{x}$ *is not.*

Section 6.1

1 Show that $\sum_{k=n}^{\infty} x^k = \frac{x^n}{1-x}$ **for** $|x| < 1$ **and** $n = 0, 1, \ldots$.

If $n \geq 1$*, then using Theorem 6.7, we have* $\sum_{k=n}^{\infty} x^k = x^n + x^{n+1} + \ldots = x^{n-1}(x^1 + x^2 + \ldots)$

$x^{n-1} \sum_{k=1}^{\infty} x^k = x^{n-1} \frac{x}{1-x} = \frac{x^n}{1-x}$.

If $n = 0$, then $\sum_{k=0}^{\infty} x^k = 1 + \sum_{k=1}^{\infty} x^k = 1 + \frac{x}{1-x} = \frac{1}{1-x}$ (also, this formula was proved in class).

2(b) Prove that $\sum_{k=1}^{\infty} \frac{(-1)^k + 4}{5^k}$ **converges and find its value.**

$$\sum_{k=1}^{\infty} \frac{(-1)^k + 4}{5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{5^k} + \sum_{k=1}^{\infty} \frac{4}{5^k} = \sum_{k=1}^{\infty} \left(-\frac{1}{5}\right)^k + 4 \sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^k = \frac{-\frac{1}{5}}{1+\frac{1}{5}} + 4 \cdot \frac{\frac{1}{5}}{1-\frac{1}{5}} = -\frac{1}{6} + 1$$

$= \frac{5}{6}$.

(The series converges because it is a linear combination of two convergent geometric series.)

5(c) Prove that $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ **diverges.**

$\sum_{k=1}^{\infty} \frac{k+1}{k^2} = \sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{k^2}\right)$. *Since* $\sum_{k=1}^{\infty} \frac{1}{k}$ *is divergent (it's the harmonic series) and each partial sum of* $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ *is larger than the corresponding partial sum of* $\sum_{k=1}^{\infty} \frac{1}{k}$*, the sequence of partial sums of* $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ *diverges.*

6(a) Prove that if $\sum_{k=1}^{\infty} a_k$ **converges, then its partial sums** s_n **are bounded.**

If $\sum_{k=1}^{\infty} a_k$*, converges then the sequence of its partial sums* $\{s_n\}$ *converges. Since every convergent sequence is bounded (Theorem 2.8),* $\{s_n\}$ *is bounded.*

6(b) Show that the converse of part (a) is false. Namely, show that a series $\sum_{k=1}^{\infty} a_k$ **may have bounded partial sums and still diverge.**

Let $a_k = (-1)^k$*. Then the sequence of partial sums of* $\sum_{k=1}^{\infty} a_k$ *is* $\{-1, 0, -1, 0, \ldots\}$*. It is bounded but divergent, so the series diverges.*