1(b) Using the method of Example 2.2, prove that \(2 \left(1 - \frac{1}{n}\right) > 2\) as \(n \to \infty\).

Given \(\epsilon > 0\), choose \(N > \frac{2}{\epsilon}\). Then for any \(n \geq N\),
\[
\left|2 \left(1 - \frac{1}{n}\right) - 2\right| = \left|\frac{2}{n}\right| = \frac{2}{n} \leq \frac{2}{N} < \epsilon.
\]

6(a) Suppose that \(\{x_n\}\) and \(\{y_n\}\) converge to the same point. Prove that \(x_n - y_n \to 0\) as \(n \to \infty\).

Suppose that \(\{x_n\}\) and \(\{y_n\}\) converge to \(a\). Then for any \(\epsilon > 0\), choose \(N_1\) such that \(\forall n \geq N_1, |x_n - a| < \frac{\epsilon}{2}\) and choose \(N_2\) such that \(\forall n \geq N_2, |y_n - a| < \frac{\epsilon}{2}\). Let \(N = \max(N_1, N_2)\). Then \(\forall n \geq N\) we have
\[
|x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |a - y_n| = |x_n - a| + |y_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

6(b) Prove that the sequence \(\{n\}\) does not converge.

Suppose that the sequence \(\{n\}\) converges. Every convergent sequence is bounded. Then \(\{n\}\) is bounded, i.e. there exists a real number \(M\) such that for any \(n \geq N\), \(M < n < M\). By Archimedean principle, for any real number \(M\) there exists a natural number \(n\) such that \(n > M\). Contradiction. Therefore our assumption that \(\{n\}\) converges was false.

6(c) Show that the converse of part (a) is false.

The converse of (a) is: if \(\{x_n\}\) and \(\{y_n\}\) are sequences such that \(x_n - y_n \to 0\) as \(n \to \infty\) then \(\{x_n\}\) and \(\{y_n\}\) converge to the same point.

Counterexample: Let \(x_n = y_n = n\), then \(x_n - y_n = 0 \to 0\), but the sequences \(\{x_n\}\) and \(\{y_n\}\) do not converge.

7(a) Let \(a\) be a fixed real number and define \(\{x_n = a\}\) for \(n \in \mathbb{N}\). Prove that the “constant” sequence \(x_n\) converges. (b) What does \(\{x_n\}\) converge to?

The constant sequence is \(a, a, a, \ldots\) We guess that this sequence converges to \(a\). Proof:

given \(\epsilon > 0\), let \(N = 1\). Then \(\forall n \geq N\) we have \(|x_n - a| = |a - a| = 0 < \epsilon\). So \(\lim_{n \to \infty} x_n = a\).

Section 2.2

5 Prove that given \(x \in \mathbb{R}\) there is a sequence \(r_n \in \mathbb{Q}\) such that \(r_n \to x\) as \(n \to \infty\).

By the density of rationals theorem, for any \(n \in \mathbb{N}\) there exists a rational number \(r_n\) such that \(x - \frac{1}{n} < r_n < x + \frac{1}{n}\). (That is, for each \(n \in \mathbb{N}\) we choose such a rational number \(r_n\), and so we get a sequence \(\{r_n\}\).) Since \(\lim_{n \to \infty} \left(x - \frac{1}{n}\right) = \lim_{n \to \infty} \left(x + \frac{1}{n}\right) = x\), by the Squeeze theorem \(\lim_{n \to \infty} r_n = x\).
3 Prove that if \( \{x_n\} \) and \( \{y_n\} \) are convergent real sequences such that \( y_n \neq 0 \) and \( \lim_{n \to \infty} y_n \neq 0 \), then \( \lim_{n \to \infty} \frac{x_n}{y_n} = \frac{\lim_{n \to \infty} x_n}{\lim_{n \to \infty} y_n} \).

Let \( n \to \infty \) \( x_n = x \) and \( n \to \infty \) \( y_n = y \). First we will prove that \( \lim_{n \to \infty} \frac{1}{y_n} = \frac{1}{y} \). Let \( \epsilon > 0 \).

Since \( y_n \to y \), \( \exists N_1 \in \mathbb{N} \) such that \( \forall n \geq N_1 \) we have \( |y_n - y| < \frac{|y|^2 \epsilon}{2} \).

Also, \( \exists N_2 \in \mathbb{N} \) such that \( \forall n \geq N_2 \) we have \( |y_n - y| < \frac{|y|}{2} \). If \( y \) is positive this implies that \( \frac{y}{2} < y_n < \frac{3y}{2} \). If \( y \) is negative then \( \frac{3y}{2} < y_n < \frac{y}{2} \). In either case, \( |y_n| > \frac{|y|}{2} \).

Let \( N = \max(N_1, N_2) \). Then \( \forall n \geq N \) we have

\[
\left| \frac{1}{y_n} - \frac{1}{y} \right| = \left| \frac{y - y_n}{y_n y} \right| = \frac{|y - y_n|}{|y_n||y|} = \frac{|y_n - y|}{|y_n||y|} < \frac{|y|^2 \epsilon}{2|y||y|} = \epsilon.
\]

Finally, \( \lim_{n \to \infty} \frac{x_n}{y_n} = \lim_{n \to \infty} \left( x_n \cdot \frac{1}{y_n} \right) = \lim_{n \to \infty} x_n \cdot \lim_{n \to \infty} \frac{1}{y_n} = x \cdot \frac{1}{y} = \frac{x}{y} \).

Another approach is to show that \( \lim_{n \to \infty} (x_n y - y_n) = 0 \) and \( \left\{ \frac{1}{y_n} \right\} \) is bounded, then

\[
\lim_{n \to \infty} \left( \frac{x_n}{y_n} - \frac{x}{y} \right) = \lim_{n \to \infty} \frac{x_n y - y_n}{y_n y} = \lim_{n \to \infty} (x_n y - y_n) \cdot \frac{1}{y_n} = 0.
\]

9 Interpret a decimal expansion \( 0.a_1a_2\ldots \) as \( 0.a_1a_2\ldots = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{a_k}{10^k} \).

Prove that (a) \( 0.5 = \frac{4999}{9} \ldots \) and (b) \( 1 = \frac{10}{9} \ldots \).

The limit on the right is the limit of the following sequence:

\[
x_1 = \sum_{k=1}^{1} \frac{a_k}{10^k} = \frac{a_1}{10^1} = 0.a_1
\]

\[
x_2 = \sum_{k=1}^{2} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} = 0.a_1a_2
\]

\[
x_3 = \sum_{k=1}^{3} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} = 0.a_1a_2a_3
\]

\[
x_4 = \sum_{k=1}^{4} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} = 0.a_1a_2a_3a_4
\]

which is the infinite decimal \( 0.a_1a_2a_3a_4 \ldots \).

(a) If \( a_1 = 4 \) and \( a_k = 9 \) for \( k \geq 2 \), we have

\[
0.4999\ldots = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{a_k}{10^k} = \frac{4}{10} + \lim_{n \to \infty} \sum_{k=2}^{n} \frac{9}{10^k} = 0.4 + \frac{1}{10} \lim_{n \to \infty} \sum_{k=1}^{n-1} \frac{9}{10^k}
\]

\[
= 0.4 + \frac{1}{10} \lim_{n \to \infty} \frac{9}{10^k} \left( \sum_{k=1}^{n-1} 1 \right) = 0.4 + \frac{1}{10} \lim_{n \to \infty} \left( 1 - \frac{1}{10^{n-1}} \right)
\]

(3) \( a \) proof of \( \lim_{n \to \infty} \frac{1}{10^n-1} = 0 \) is similar to ex. 2 on p. 36

\[
= 0.4 + \frac{1}{10} \cdot 1 = 0.5.
\]

(b) \( 0.9999\ldots = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{9}{10^k} = \lim_{n \to \infty} \frac{10 - 1}{10^k} \) (use exercise 1(c) on p. 17)

\[
= \lim_{n \to \infty} \left( 1 - \frac{1}{10^n} \right) = 1.
\]