Math 171 Solutions to homework problems Spring 2005

Section 2.3

1 Prove that $x_n = \dfrac{(n^2+20n+35)\sin(n^3)}{n^2+n+1}$ has convergent a subsequence.

$\displaystyle\lim_{n\to\infty} \frac{n^2+20n+35}{n^2+n+1} = \lim_{n\to\infty} \frac{1+\frac{20}{n}+\frac{35}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}} = 1$, *and every convergent sequence is bounded, therefore* $\left\{ y_n = \dfrac{n^2+20n+35}{n^2+n+1} \right\}$ *is bounded, so there exists* $M \in \mathbb{R}$ *such that* $|y_n| \le M$ *for all* $n \in \mathbb{N}$. *Also,* $\{z_n = \sin(n^3)\}$ *is bounded:* $|z_n| \le 1$ *for all* $n \in \mathbb{N}$. *Therefore* $\{x_n\}$ *is bounded:* $|x_n| = |y_n z_n| = |y_n||z_n| \le M$ *for all* $n \in \mathbb{N}$. *Every bounded sequence has a convergent subsequence, thus* $\{x_n\}$ *has a convergent subsequence.*

2 Suppose that $E \in \mathbb{R}$ is a nonempty bounded set and $\sup E \notin E$. Prove that there exists a strictly increasing sequence $\{x_n\}$ that converges to $\sup E$ such that $x_n \in E$ for all $n \in \mathbb{N}$.

First choose any element $x_1 \in E$. *Since* $\sup E \notin E$, $x_1 < \sup E$. *By the Approximation Property for suprema, for each* $n \ge 2$ *there exists* $x_n \in E$ *such that*
$\max(\sup E - \frac{1}{n}, x_{n-1}) < x_n < \sup E$. *Then we get a sequence such that* $x_1 < x_2 < x_3 < \ldots$, *i.e. the sequence* $\{x_n\}$ *is strictly increasing. Also we have* $\sup E - \frac{1}{n} < x_n < \sup E$. *Since* $\displaystyle\lim_{n\to\infty} \sup E - \frac{1}{n} = \lim_{n\to\infty} \sup E = \sup E$, *by the Squeeze theorem* $\displaystyle\lim_{n\to\infty} x_n = \sup E$.

Section 2.4

1 Prove (without using Theorem 2.29) that the sum of two Cauchy sequences is Cauchy.

Let $\{x_n\}$ *and* $\{y_n\}$ *be Cauchy.*
Then for any $\epsilon > 0$ *there exists* $N_1 \in \mathbb{N}$ *such that for any* $n, m \ge N_1$, $|x_n - x_m| < \frac{\epsilon}{2}$,
and there exists $N_2 \in \mathbb{N}$ *such that for any* $n, m \ge N_2$, $|y_n - y_m| < \frac{\epsilon}{2}$.
Let $N = \max(N_1, N_2)$. *Then for any* $n, m \ge N$,
$|(x_n + y_n) - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)| \le |x_n - x_m| + |y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Thus $\{x_n + y_n\}$ *is Cauchy.*

2 Prove that if $\{x_n\}$ is a sequence that satisfies $|x_n| \le \dfrac{1+n}{1+n+2n^2}$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy.

We have $-\dfrac{1+n}{1+n+2n^2} \le x_n \le \dfrac{1+n}{1+n+2n^2}$.

$\displaystyle\lim_{n\to\infty} \frac{1+n}{1+n+2n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n}+2} = 0$, *and similarly* $\displaystyle\lim_{n\to\infty} -\frac{1+n}{1+n+2n^2} = 0$. *Therefore by the Squeeze theorem* $\displaystyle\lim_{n\to\infty} x_n = 0$, *so* $\{x_n\}$ *is convergent. Then it is Cauchy.*

3 Suppose that $x_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. If $\{x_n\}$ is Cauchy, prove that there are numbers a and N such that $x_n = a$ for all $n \ge N$.

Since $\{x_n\}$ *is Cauchy, there exists* $N \in \mathbb{N}$ *such that for all* $n, m \ge N$, $|x_n - x_m| < 1$ *(use the definition with* $\epsilon = 1$*). In particular (if* $m = N$*), for all* $n \ge N$, $|x_n - x_N| < 1$. *Let* $a = x_N$. *Then we have that for all* $n \ge N$, $|x_n - a| < 1$. *Since both* x_n *and* a *are integers,* $x_n = a$.

Section 3.1

1(a) Using Definition 3.1, prove that $\lim_{x\to 2} x^2 - x + 1 = 3$.

Given $\epsilon > 0$, *let* $\delta = min\left(1, \frac{\epsilon}{4}\right)$. *Then for any* x *such that* $0 < |x-2| < \delta$, *we have* $|x-2| < 1$ *and* $|x-2| < \frac{\epsilon}{4}$. *The first inequality implies that* $-1 < x-2 < 1$, *so* $2 < x+1 < 4$, *so* $|x+1| < 4$ *(see below why we need this).*

So for any x *such that* $0 < |x-2| < \delta$, *we have*
$|x^2 - x + 1 - 3| = |x^2 - x - 2| = |(x-2)(x+1)| = |x-2||x+1| < \frac{\epsilon}{4} \cdot 4 = \epsilon$.

3(d) Evaluate the limit $\lim_{x\to 1} \frac{x^n - 1}{x-1}$, $n \in \mathbb{N}$, using results from this section.

$$\lim_{x\to 1} \frac{x^n - 1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^{n-1} + x^{n-2} + \ldots + x + 1)}{x-1} = \lim_{x\to 1}(x^{n-1} + x^{n-2} + \ldots + x + 1) = n.$$

5 Prove Theorem 3.9: Suppose that $a \in \mathbb{R}$, I is an open interval that contains a, and f, g, h are real functions defined everywhere on I except possibly at a, then
(i) If $g(x) \le h(x) \le f(x)$ for all $x \in I\backslash\{a\}$, and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, then the limit of $h(x)$ exists, as $x \to a$, and $\lim_{x\to a} h(x) = L$.
(ii) If $|g(x)| \le M$ for all $x \in I\backslash\{a\}$ and $f(x) \to 0$ as $x \to a$, then $\lim_{x\to a} f(x)g(x) = 0$.

(i) Since $\lim_{x\to a} f(x) = L$, *by the sequential characterization for limits, for any sequence* x_n *converging to* a *and such that* $x_n \in I\backslash\{a\}$, $\lim_{n\to\infty} f(x_n) = L$.
Since $\lim_{x\to a} g(x) = L$, *by the sequential characterization for limits, for any sequence* x_n *converging to* a *and such that* $x_n \in I\backslash\{a\}$, $\lim_{n\to\infty} g(x_n) = L$.
Since $g(x_n) \le h(x_n) \le f(x_n)$ *for all* $n \in \mathbb{N}$ *,by the Squeeze theorem for sequences we have* $\lim_{n\to\infty} h(x_n) = L$. *Then by the sequential characterization for limits again,* $\lim_{x\to a} h(x) = L$.

(ii) Since $\lim_{x\to a} f(x) = 0$, *by the sequential characterization for limits, for any sequence* x_n *converging to* a *and such that* $x_n \in I\backslash\{a\}$, $\lim_{n\to\infty} f(x_n) = 0$.
Since $|g(x_n)| \le M$ *for all* $n \in \mathbb{N}$, *by the second part of the Squeeze theorem for sequences we have* $\lim_{n\to\infty} f(x_n)g(x_n) = 0$. *Then by the sequential characterization for limits again,* $\lim_{x\to a} f(x)g(x) = 0$.