Math 171 **Solutions to homework problems** **Spring 2005**

Section 3.2

1 Using definitions (rather than limit theorems) prove that $\lim_{x\to a^+} f(x)$ exists and equals L in each of the following cases.

(a) $f(x) = \frac{|x|}{x}$, $a = 0$, and $L = 1$.

Given $\varepsilon > 0$, set $\delta = 1$. Then for any $x \in (0, \delta)$ we have $x > 0$, therefore $|x| = x$, so $\left|\frac{|x|}{x} - 1\right| = \left|\frac{x}{x} - 1\right| = 0 < \varepsilon$.

(d) $f(x) = \frac{1}{x^2-1}$, $a = 1$, and $L = \infty$.

Let $M \in \mathbb{R}$ be given. If $M > 0$, set $\delta = min\left(1, \frac{1}{3M}\right)$. Then for any $x \in (1, 1+\delta)$ we have $1 < x < 2$ and $1 < x < 1 + \frac{1}{3M}$. The first inequality implies that $2 < x + 1 < 3$. The second inequality implies that $0 < x-1 < \frac{1}{3M}$. Therefore $0 < x^2-1 = (x+1)\cdot(x-1) < 3\cdot\frac{1}{3M} = \frac{1}{M}$. Then $\frac{1}{x^2-1} > M$.
If $M \le 0$, then by the above proof there exists δ such that for any $x \in (1, 1+\delta)$, $\frac{1}{x^2-1} > 1 > M$.

3(e) Evaluate the limit if it exists: $\lim_{x\to\infty} \frac{\sin x}{x^2}$.

Since $\frac{\sin x}{x^2} = \sin x \cdot \frac{1}{x^2}$, $\sin x$ is bounded, and $\lim_{x\to\infty} \frac{1}{x^2} = 0$, by the second part of the Squeeze theorem $\lim_{x\to\infty} \frac{\sin x}{x^2} = 0$.

4 Recall that a polynomial of degree n is a function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, n$ and $a_n \neq 0$.

(a) Prove that $\lim_{x\to a} x^n = a^n$ for $n = 0, 1, \ldots$.

Proof by induction.
Basis step. If $n = 0$, $\lim_{x\to a} x^0 = \lim_{x\to a} 1 = 1^0$.
Inductive step. Suppose that $\lim_{x\to a} x^k = a^k$ is true, then
$\lim_{x\to a} x^{k+1} = \lim_{x\to a} x^k \cdot x = \lim_{x\to a} x^k \cdot \lim_{x\to a} x = a^k \cdot a = a^{k+1}$.

(b) Prove that if P is a polynomial, then $\lim x \to aP(x) = P(a)$ for every $a \in \mathbb{R}$.

Proof by (strong) induction.
Basis step. If $n = 0$, then $P(x) = a_0$, and $\lim_{x\to a} P(x) = \lim_{x\to a} a_0 = a_0 = P(a)$.
Inductive step. Suppose that the statement is true for any polynomial of degree less than or equal to k. We want to prove the statement is true for any polynomial of degree $k+1$. Let $P(x) = a_{k+1}x^{k+1} + a_k x^k + \ldots + a_1 x + a_0$ and let $Q(x) = a_k x^k + \ldots + a_1 x + a_0$. Then the statement is true for $Q(x)$. Therefore $\lim_{x\to a} P(x) = \lim_{x\to a}(a_{k+1}x^{k+1} + Q(x)) = a_{k+1}a^{k+1} + Q(a) = P(a)$.

(c) Suppose that P is a polynomial and $P(a) > 0$. Prove that $\dfrac{P(x)}{x-a} \to \infty$ as $x \to a^+$, $\dfrac{P(x)}{x-a} \to -\infty$ as $x \to a^-$, but $\lim_{x\to a} \dfrac{P(x)}{x-a}$ does not exist.

Part 1. Let $M \in \mathbb{R}$ be given. Without loss of generality (see problem 1(d)) we can assume that $M > 0$.
Since $P(a) > 0$, by the proof of the sign-preserving property there exists $\delta > 0$ such that $a < x < a + \delta$ implies $P(x) > \dfrac{P(a)}{2}$. Let $\delta' = min\left(\delta, \dfrac{P(a)}{2M}\right)$.

Then $a < x < a + \delta'$ implies that $0 < x - a < \delta'$, therefore $\dfrac{P(x)}{x-a} > \dfrac{P(a)}{2(x-a)} > \dfrac{P(a)}{2\delta'} \geq \dfrac{P(a)}{2\frac{P(a)}{2M}} = M$.

Part 2. Let $M \in \mathbb{R}$ be given. Without loss of generality (similarly to above) we can assume that $M < 0$.
As in part 1, there exists $\delta > 0$ such that $a - \delta < x < a$ implies $P(x) > \dfrac{P(a)}{2}$.
Let $\delta' = min\left(\delta, -\dfrac{P(a)}{2M}\right)$. Then $\delta' \leq -\dfrac{P(a)}{2M}$.

Then $a - \delta' < x < a$ implies that $x - a < 0$, therefore $\dfrac{P(x)}{x-a} < \dfrac{P(a)}{2(x-a)}$.

Also, $x - a > -\delta' \geq \dfrac{P(a)}{2M}$, so $\dfrac{1}{x-a} < \dfrac{1}{\frac{P(a)}{2M}}$. It follows that $\dfrac{P(x)}{x-a} < \dfrac{P(a)}{2(x-a)} < \dfrac{P(a)}{-2\delta} \leq \dfrac{P(a)}{2\frac{P(a)}{2M}} = M$.

Part 3. Since $\lim_{x\to a^+} \dfrac{P(x)}{x-a}$ and $\lim_{x\to a^-} \dfrac{P(x)}{x-a}$ are not equal, $\lim_{x\to a} \dfrac{P(x)}{x-a}$ does not exist.

Section 3.3

5 Show that there exist nowhere continuous functions f and g whose sum $f + g$ is continuous on $\mathbb{R}$. Show that the same is true for the product of functions.

Example: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ and $g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$.

Then $f(x) + g(x) = 1$ and $f(x)g(x) = 0$ for all $x \in \mathbb{R}$.