1 Using definitions (rather than limit theorems) prove that $\lim_{x \to a^+} f(x)$ exists and equals $L$ in each of the following cases.

(a) $f(x) = \frac{|x|}{x}$, $a = 0$, and $L = 1$.

Given $\varepsilon > 0$, set $\delta = 1$. Then for any $x \in (0, \delta)$ we have $x > 0$, therefore $|x| = x$, so
\[ \left| \frac{|x|}{x} - 1 \right| = \left| \frac{x}{x} - 1 \right| = 0 < \varepsilon. \]

(b) $f(x) = \frac{1}{x^2 - 1}$, $a = 1$, and $L = \infty$.

Let $M \in \mathbb{R}$ be given. If $M > 0$, set $\delta = \min \left(1, \frac{1}{3M} \right)$. Then for any $x \in (1, 1 + \delta)$ we have
\[ 1 < x < 2 \quad \text{and} \quad 1 < x - 1 < \frac{1}{3M}. \]
The first inequality implies that $2 < x + 1 < 3$. The second inequality implies that $0 < x-1 < \frac{1}{3M}$. Therefore $0 < x^2 - 1 = (x+1)(x-1) < 3 \cdot \frac{1}{3M} = \frac{1}{M}$. Then $\frac{1}{x^2 - 1} > M$.

If $M \leq 0$, then by the above proof there exists $\delta$ such that for any $x \in (1, 1 + \delta)$, $\frac{1}{x^2 - 1} > 1 > M$.

3(e) Evaluate the limit if it exists:
\[ \lim_{x \to \infty} \frac{\sin x}{x^2}. \]

Since $\frac{\sin x}{x^2} = \sin x \cdot \frac{1}{x^2}$, $\sin x$ is bounded, and \[ \lim_{x \to \infty} \frac{1}{x^2} = 0, \] by the second part of the Squeeze theorem \[ \lim_{x \to \infty} \frac{\sin x}{x^2} = 0. \]

4 Recall that a polynomial of degree $n$ is a function of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, n$ and $a_n \neq 0$.

(a) Prove that $\lim_{x \to a} x^n = a^n$ for $n = 0, 1, \ldots$.

Proof by induction.
Basis step. If $n = 0$, $\lim_{x \to a} x^0 = \lim_{x \to a} 1 = 1^0$.
Inductive step. Suppose that $\lim_{x \to a} x^k = a^k$ is true, then
\[ \lim_{x \to a} x^{k+1} = \lim_{x \to a} x^k \cdot x = \lim_{x \to a} x^k \cdot \lim_{x \to a} x = a^k \cdot a = a^{k+1}. \]

(b) Prove that if $P$ is a polynomial, then $\lim_{x \to a} P(x) = P(a)$ for every $a \in \mathbb{R}$.

Proof by (strong) induction.
Basis step. If $n = 0$, then $P(x) = a_0$, and $\lim_{x \to a} P(x) = \lim_{x \to a} a_0 = a_0 = P(a)$.
Inductive step. Suppose that the statement is true for any polynomial of degree less than or equal to $k$. We want to prove the statement is true for any polynomial of degree $k + 1$. Let $P(x) = a_{k+1} x^{k+1} + a_k x^k + \ldots + a_1 x + a_0$ and let $Q(x) = a_k x^k + \ldots + a_1 x + a_0$. Then the statement is true for $Q(x)$. Therefore $\lim_{x \to a} P(x) = \lim_{x \to a} (a_{k+1} x^{k+1} + Q(x)) = a_{k+1} a^{k+1} + Q(a) = P(a)$. 

(c) Suppose that $P$ is a polynomial and $P(a) > 0$. Prove that $\frac{P(x)}{x - a} \to -\infty$ as $x \to a^-$, but $\lim_{x \to a^-} \frac{P(x)}{x - a}$ does not exist.

Part 1. Let $M \in \mathbb{R}$ be given. Without loss of generality (see problem 1(d)) we can assume that $M > 0$.

Since $P(a) > 0$, by the proof of the sign-preserving property there exists $\delta > 0$ such that $a < x < a + \delta$ implies $P(x) > \frac{P(a)}{2}$. Let $\delta' = \min\left(\delta, \frac{P(a)}{2M}\right)$.

Then $a < x < a + \delta'$ implies that $0 < x - a < \delta'$, therefore $\frac{P(x)}{x - a} > \frac{P(a)}{2(x - a)} > \frac{P(a)}{2\delta'} \geq \frac{P(a)}{2\delta' M} = M$.

Part 2. Let $M \in \mathbb{R}$ be given. Without loss of generality (similarly to above) we can assume that $M < 0$.

As in part 1, there exists $\delta > 0$ such that $a - \delta < x < a$ implies $P(x) > \frac{P(a)}{2}$. Let $\delta' = \min\left(\delta - \frac{P(a)}{2M}\right)$. Then $\delta' \leq -\frac{P(a)}{2M}$.

Then $a - \delta' < x < a$ implies that $x - a < 0$, therefore $\frac{P(x)}{x - a} < \frac{P(a)}{2(x - a)}$.

Also, $x - a > -\delta' \geq \frac{P(a)}{2M}$, so $\frac{1}{x - a} < \frac{1}{\frac{P(a)}{2M}}$. It follows that $\frac{P(x)}{x - a} < \frac{P(a)}{2(x - a)} < \frac{P(a)}{-2\delta} \leq \frac{P(a)}{2\delta' M} = M$.

Part 3. Since $\lim_{x \to a^+} \frac{P(x)}{x - a}$ and $\lim_{x \to a^-} \frac{P(x)}{x - a}$ are not equal, $\lim_{x \to a} \frac{P(x)}{x - a}$ does not exist.

Section 3.3

5 Show that there exist nowhere continuous functions $f$ and $g$ whose sum $f + g$ is continuous on $\mathbb{R}$. Show that the same is true for the product of functions.

Example: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ and $g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$.

Then $f(x) + g(x) = 1$ and $f(x)g(x) = 0$ for all $x \in \mathbb{R}$. 