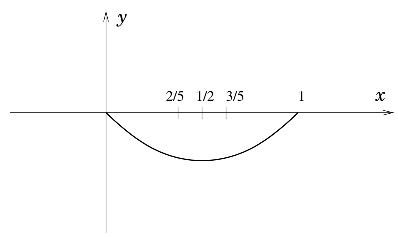
# Math 171 — Solutions to homework problems — Spring 2005

## Section 5.1

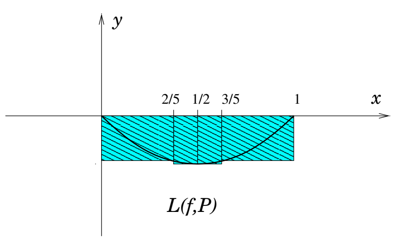
**1(c) For $f(x) = x^2 - x$, compute $U(f,P)$, $L(f,P)$, and $\int_0^1 f(x)dx$, where $P = \left\{0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1\right\}$. Find out whether the lower sum or the upper sum is a better approximation to the integral. Graph $f$ and explain why this is so.**

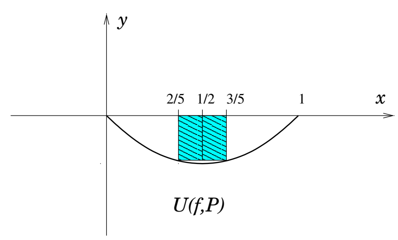



$$U(f,P) = f(0)\frac{2}{5} + f\left(\frac{2}{5}\right)\frac{1}{10} + f\left(\frac{3}{5}\right)\frac{1}{10} + f(1)\frac{2}{5} = -0.048.$$

$$L(f,P) = f\left(\frac{2}{5}\right)\frac{2}{5} + f\left(\frac{1}{2}\right)\frac{1}{10} + f\left(\frac{1}{2}\right)\frac{1}{10} + f\left(\frac{3}{5}\right)\frac{2}{5} = -0.242.$$

$$\int_0^1 f(x)dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} = -0.1\overline{6}.$$




*The lower sum is a better approximation. We see from the graphs that the function is concave upward, and so for each subinterval, the difference between the area of the lower approximating rectangle and the area of the region above the curve is less than the difference between the area of the upper approximating rectangle and the area of the region above the curve.*

**2(c($\beta$)) Let $P_n = \left\{\frac{j}{n} : \; j = 0, 1, \ldots, n\right\}$. Use Exercise 1, p. 17, to find formulas for the upper and lower sums of $f(x) = x^2$ on $P_n$, and use them to compute the value of $\int_0^1 f(x)dx$.**

$$U(f,P_n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

$$L(f,P_n) = \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)\frac{1}{n} = \sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2 = \frac{1}{n^3}\sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6n^3}$$

$$= \frac{(n-1)(2n-1)}{6n^2}.$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{2}{6} = \frac{1}{3}.$$

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to\infty} \frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6} = \frac{2}{6} = \frac{1}{3}.$$

*Since* $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = \frac{1}{3}$, $\int_0^1 f(x)dx = \frac{1}{3}$.

**4 Suppose that $[a,b]$ is a closed, nondegenerate interval and $f : [a,b] \to \mathbb{R}$ is bounded.**
**(a) Prove that if $f$ is continuous at $x_0 \in [a,b]$ and $f(x_0) \neq 0$, then $(L)\int_a^b |f(x)|dx > 0$.**

*Since $f(x_0) \neq 0$, $|f(x_0)| > 0$, and by the sign-preserving property there exist $\delta > 0$ and $\varepsilon > 0$ such that $|f(x)| > \varepsilon$ for $x \in (x_0-\delta, x_0+\delta)$. Let $P$ be a partition such that one of its subintervals is contained in $(x_0-\delta, x_0+\delta)$. Then the area of the lower approximating rectangle over that subinterval is positive, and the areas of all other lower approximating rectangles is nonnegative, therefore the lower Riemann sum over $P$ is positive. Then $(L)\int_a^b |f(x)|dx > 0$.*

**(b) Show that if $f$ is continuous on $[a,b]$, then $\int_a^b |f(x)|dx = 0$ if and only if $f(x) = 0$ for all $x \in [a,b]$.**

($\Rightarrow$) *If there exists $x_0 \in [a,b]$ such that $f(x_0) \neq 0$ then by part (a) $(L)\int_a^b |f(x)|dx > 0$, and therefore $\int_a^b |f(x)|dx \neq 0$.*

($\Leftarrow$) *If $f(x) = 0$ for all $x$ then $\int_a^b |f(x)|dx = \int_a^b 0 = 0$.*

**(c) Does part (b) hold if the absolute values are removed? If it does, prove it. If it does not, provide a counterexample.**

*No. Counterexample: $\int_{-1}^1 xdx = 0$ but $f(x) = x \neq 0$.*

**7(a) Prove that $(U)\int_a^b (f(x)+g(x))dx \leq (U)\int_a^b f(x)dx + (U)\int_a^b g(x)dx$ and $(L)\int_a^b (f(x)+g(x))dx \leq (L)\int_a^b f(x)dx + (L)\int_a^b g(x)dx$.**

*Let $P$ be any partition. Then for each subinterval and for all $x$ in that subinterval, $(f+g)(x) = f(x)+g(x) \leq M_i(f)+M_i(g)$, therefore $M_i(f+g) \leq M_i(f)+M_i(g)$. Multiplying both sides of this inequality by $(x_i - x_{i-1})$ and taking the sum over $i$ from 1 to $n$ gives $U(f+g,P) \leq U(f,P)+U(g,P)$. Taking inf of both sides over $P$ gives the first inequality.*

*Similarly, $(f+g)(x) = f(x)+g(x) \geq m_i(f)+m_i(g)$, therefore $m_i(f+g) \geq m_i(f)+m_i(g)$. Multiplying both sides of this inequality by $(x_i - x_{i-1})$ and taking the sum over $i$ from 1 to $n$ gives $L(f+g,P) \geq L(f,P)+L(g,P)$. Taking sup of both sides over $P$ gives the second inequality.*