**Math 171** **Spring 2005**

# Final Exam

## Part I. Take home

The following problems are **due Wednesday, May 18, at 8:30 AM**. Each problem is worth 10 points. Show all your work. An answer without explanation will receive no credit. Provide detailed proofs.

You are allowed to use your notes, books, or any printed or posted on the Internet materials. However, **you may not discuss these problems with your classmates, friends, professors, or anybody else**. You may not borrow anybody's notes concerning these problems or give your notes to anybody. This requrement is very strict. If you break this rule then this part of your final will not be counted.

1. Solve the inequality: $x^3 + 4x^2 + 4x \leq 0$.

2. Let $A, B \subset \mathbb{R}$ and $E = A \cup B$. Prove that if $E$ has a supremum and both $A$ and $B$ are nonempty, then sup$A$ and sup$B$ exist, and sup$E$ is one of the numbers sup$A$ or sup$B$.

3. Let $\{b_n\}$ be a sequence of nonnegative numbers that converge to 0, $c > 0$, and suppose that $\{x_n\}$ is a sequence such that $|x_n - a| \leq b_n c$ for all $n$. Prove that $\{x_n\}$ converges to $a$.

4. Prove that $x^2 \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, 1)$.

5. If $f(x)$ is continuous on $[a, b]$ and there exist numbers $\alpha \neq \beta$ such that
$$\alpha \int_a^x f(t)dt + \beta \int_x^b f(t)dt = 0$$
holds for all $x \in (a, b)$, prove that $f(x) = 0$ for all $x \in [a, b]$.
(Hint: use the Fundamental Theorem of Calculus.)

## Part II. In class, May 18, 7:30-8:30 AM

"Train hard, fight easy"
Alexander V. Suvorov, Russian Field Marshal, 1729-1800

This part will consist of 5 questions:

1. Give a definition.
   You should know the following definitions:
   (a) Bounded above, upper bound, supremum of a set (p. 18); bounded below, lower bound, infimum of a set (p. 21)
   (b) One-to-one and onto function (p. 25)
   (c) Image and inverse image of a set under a function (p. 32)
   (d) Convergent sequence (p. 35)
   (e) Subsequence (p. 37)
   (f) Bounded above, bounded below, bounded sequence (p. 37)
   (g) Divergent to $\infty$ or $-\infty$ sequence (p. 41)
   (h) Increasing, strictly increasing, decreasing, strictly decreasing, monotone sequence (p. 45)
   (i) Cauchy sequence (p. 49)
   (j) Limit of a function at a point (p. 58)
   (k) Right-hand limit and left-hand limit of a function (p.66)
   (l) Limit at infinty (as $x \to \infty$ or $x \to -\infty$), infinite limit ($f(x) \to \infty$ or $f(x) \to -\infty$) (p. 67)
   (m) Continuous function (p. 72)
   (n) Composition of functions (p. 73)
   (o) Bounded function (p. 73)
   (p) Uniformly continuous function (p. 80)
   (q) Differentiable function (p. 85)
   (r) Increasing, strictly increasing, decreasing, strictly decreasing, monotone, strictly monotone function (p. 102)
   (s) Partition of an interval; norm of a partition; refinement of a partition (p. 107)
   (t) Upper Riemann sum; lower Riemann sum (p. 108)
   (u) Riemann integrable function (p. 110)
   (v) Upper integral; lower integral; integral (p. 112)
   (w) Riemann sum; convergent Riemann sums (p. 117)
   (x) Locally integrable function; improperly integrable function (p. 137)
   (y) Infinite series; partial sum of a series; convergent (divergent) series (p. 154-155)
   (z) Absolutely convergent series; conditionally convergent series (p. 165)

2. Give an example.
   Practice questions:
   (a) Give an example of a divergent sequence that has a convergent subsequence.
   (b) Give an example of a sequence that has no convergent subsequences.
   (c) Can you give an example of sequence that is Cauchy but not convergent?
   (d) Give an example of a function $f(x)$ defined everywhere on $\mathbb{R}$ but such that $\lim_{x\to 0} f(x)$ does not exist.
   (e) Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous everywhere but is not differentiable at at least one point.
   (f) Give an example of a function from $[0, 1]$ to $\mathbb{R}$ that is not integrable.
   (g) Give an example of a conditionally convergent series.

3. Question.
   Practice questions:
   (a) Does the set $\{\sin(n) \mid n \in \mathbb{Z}\}$ have a supremum? If so, what is it?
   (b) Is the function $f(x) : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$ onto? Is it one-to-one?
   (c) Is the sequence $\{n\}$ bounded above? Is it bounded below? Is it bounded?
   (d) Let $[x]$ denote the integer part of $x$. Is the sequence $\{\left[\frac{n}{2}\right]\}$ increasing?
   (e) Is the function $f(x) = \sqrt{x}$ continuous on $[0, +\infty)$? Is it uniformly continuous?
   (f) Is the function $\dfrac{1}{x}$ improperly integrable on $(-1, 1)$? If so, what is the value of $\displaystyle\int_{-1}^{1} \frac{1}{x} dx$?
   (g) Is the series $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ convergent? Is it absolutely convergent?

4. State a postulate or a theorem.
   You should know the following statements:
   (a) Field axioms (p. 2)
   (b) Order axioms (p. 4)
   (c) Well-ordering principle (p. 13)
   (d) Completeness axiom (p. 19)
   (e) Density of rationals (p. 20)
   (f) Squeeze theorem for sequences (p. 39)
   (g) Comparison theorem (p. 43)
   (h) Monotone convergence theorem (p. 45)
   (i) Nested interval property (p. 46)
   (j) Bolzano-Weierstrass theorem (p. 47)

(k) Cauchy theorem (p. 50)
(l) Sequential characterization of limits (p. 60)
(m) Sequential characterization of continuity (Theorem 3.21, p. 72)
(n) Extreme value theorem (p. 74)
(o) Intermediate value theorem (p. 75)
(p) The derivative of the sum, constant multiple, product, and quotient of functions (p. 92)
(q) Chain rule (p. 92)
(r) Rolle's theorem (p. 94)
(s) Inverse function theorem (p. 103)
(t) Linear property of Riemann integral (p. 119)
(u) Comparison theorem for integrals (p. 121)
(v) Fundamental theorem of Calculus (p. 127)
(w) Convergence of a geometric series (p. 156)
(x) $p$-series test (p. 162)
(y) Comparison test (p. 162)
(z) Alternating series test (p. 175)

5. State and prove a theorem.
   You should know the statements and proofs of the following theorems:
   (a) Approximation property for suprema (p. 18)
   (b) Sign-preserving property (p. 75)
   (c) Mean value theorem (Theorem 4.15 (ii), p. 96; either see the proof given in class or state and prove part (i) first.)
   (d) Increasing / decreasing test (Theorem 4.24, p. 102)
   (e) Integration by parts (p. 129)
   (f) Divergence test (p. 156)