1. Give the definition of \( \lim_{x \to -\infty} f(x) = L \).

   Let \( f \) be defined on some interval \((c, +\infty)\). We say that \( \lim_{x \to -\infty} f(x) = L \) if for any \( \varepsilon > 0 \) there exists an \( M \in \mathbb{R} \) such that \( x > M \) implies \( |f(x) - L| < \varepsilon \).

2. State Rolle's theorem.

   Let \( a < b \). If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( f(a) = f(b) \), then there exists a point \( c \in (a, b) \) such that \( f'(c) = 0 \).

3. State and prove the sign-preserving property.

   Let \( I \) be an open (nondegenerate) interval. If \( f(x) \) is continuous at a point \( a \in I \) and \( f(a) > 0 \), then there exist positive numbers \( \varepsilon \) and \( \delta \) such that for \( x \in I \), \( |x - a| < \delta \) implies \( f(x) > \varepsilon \).

   **Proof.** Let \( \varepsilon = \frac{f(a)}{2} \). Since \( f(x) \) is continuous at \( a \), there exists \( \delta > 0 \) such that \( |x - a| < \delta \) implies \( |f(x) - f(a)| < \varepsilon \). Then \( |f(x) - f(a)| < \frac{f(a)}{2} \), so \( f(x) > f(a) < \frac{f(a)}{2} \).

   Adding \( f(a) \) gives \( \frac{f(a)}{2} < f(x) < \frac{3f(a)}{2} \), so we have \( f(x) > \frac{f(a)}{2} = \varepsilon \).

4. Find all \( a \in \mathbb{R} \) such that \( f(x) = \frac{ax + 2}{x + 1} \) is strictly increasing on \((1, 2)\).

   \[
   f'(x) = \frac{a(x + 1) - (ax + 2)}{(x + 1)^2} = \frac{a - 2}{(x + 1)^2}.
   \]

   If \( f(x) \) is strictly increasing then \( f'(x) \geq 0 \), so we have \( a - 2 > 0 \), or \( a \geq 2 \).

   If \( a > 2 \) then we have \( f'(x) > 0 \), so \( f(x) \) is strictly increasing in its domain (which contains the interval \((1, 2)\)).

   If \( a = 2 \), \( f(x) = \frac{2x + 2}{x + 1} = 2 \) is not strictly increasing.

   **Answer:** \( a > 2 \).

5. Let \( f(x) \) and \( g(x) \) be uniformly continuous on \( \mathbb{R} \). Prove that \( (f + g)(x) \) is uniformly continuous on \( \mathbb{R} \).

   Let \( \varepsilon > 0 \).

   Since \( f(x) \) is uniformly continuous on \( \mathbb{R} \), there exists \( \delta_1 > 0 \) such that for any \( x_1, x_2 \in \mathbb{R} \), \( |x_1 - x_2| < \delta_1 \) implies \( |f(x_1) - f(x_2)| < \frac{\varepsilon}{2} \).

   Since \( g(x) \) is uniformly continuous on \( \mathbb{R} \), there exists \( \delta_2 > 0 \) such that for any \( x_1, x_2 \in \mathbb{R} \), \( |x_1 - x_2| < \delta_2 \) implies \( |g(x_1) - g(x_2)| < \frac{\varepsilon}{2} \).

   Let \( \delta = \min(\delta_1, \delta_2) \). Then for any \( x_1, x_2 \in \mathbb{R} \) such that \( |x_1 - x_2| < \delta \) we have

   \[
   |(f + g)(x_1) - (f + g)(x_2)| = |f(x_1) + g(x_1) - f(x_2) - g(x_2)| = |f(x_1) - f(x_2) + g(x_1) - g(x_2)|
   \]

   \[
   \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
   \]

   Thus \( (f + g)(x) \) is uniformly continuous on \( \mathbb{R} \).

6. Prove or disprove each of the following statements:

   (a) If a function is continuously differentiable on \( \mathbb{R} \) then it is twice differentiable on \( \mathbb{R} \).

   The statement is false. Let \( f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases} \). Then \( f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases} \) is continuous but not differentiable.

   (b) If a function is continuously differentiable 100 times on \( \mathbb{R} \) then it is differentiable 101 times on \( \mathbb{R} \).

   The statement is false. Let \( f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{101} & \text{if } x \geq 0 \end{cases} \). Then \( f^{(100)}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 101!x & \text{if } x \geq 0 \end{cases} \) is continuous but not differentiable.