Math 75A

Practice test 2 - Solutions

Multiple choice questions: circle the correct answer

1. Solve for \( x \): \( \log_{\frac{1}{2}} x = 3 \).

A. 6          B. \( \frac{1}{6} \)          C. 8          D. \( \frac{1}{8} \)
E. None of the above

(because \( \left(\frac{1}{2}\right)^3 = \frac{1}{8} \))

2. How many vertical asymptotes does the curve \( y = \frac{x + 1}{x(x + 2)(x + 3)} \) have?

A. 0          B. 1          C. 2          D. 3          E. 4

\( x = 0, x = -2, \) and \( x = -3 \)

3. \( \lim_{x \to 2} \frac{5}{x - 2} = \)

A. 0          B. 5          C. \( \infty \)          D. \( -\infty \)          E. Does not exist

(because the limits from the right and from the left are not equal)

4. \( \lim_{x \to -\infty} \frac{x + 2}{3x + 4} = \)

A. 1          B. \( \frac{1}{3} \)          C. \( \frac{1}{3} \)          D. 0          E. Does not exist

(divide the numerator and denominator by \( x \))

5. The function \( f(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases} \) is

A. continuous everywhere
B. continuous at 1 but discontinuous at \(-1\)
C. continuous at \(-1\) but discontinuous at 1
D. continuous at all points except for 1 and \(-1\)
E. discontinuous everywhere

(because there is a break in the graph at 1 but not at \(-1\))

6. Find the rate of change of \( y = 3x + 5 \) at \( x = 4 \).

A. 3          B. 4          C. 5          D. 17          E. None of the above

(the rate of change is \( y'(4) \), the slope of the graph, which is 3)

7. Find the equation of the line tangent to the curve \( y = x^2 \) at \( (2, 4) \).

A. \( y = 4x \)          B. \( y = 4x - 4 \)          C. \( y = 4x + 4 \)          D. \( y = -4x \)          E. \( y = -4x - 4 \)

(first find the slope of the tangent line: \( y'(2) \); then use \( y - 4 = y'(2)(x - 2) \))
Regular problems: show all your work

8. Solve the following equations:

(a) \( \ln(5x - 2) = 3 \)
\[ e^{\ln(5x - 2)} = e^3 \]
\[ 5x - 2 = e^3 \]
\[ 5x = e^3 + 2 \]
\[ x = \frac{e^3 + 2}{5} \]

(b) \( e^{3t+1} = 100 \)
\[ \ln(e^{3t+1}) = \ln 100 \]
\[ 3t + 1 = \ln 100 \]
\[ 3t = \ln 100 - 1 \]
\[ t = \frac{\ln 100 - 1}{3} \]

(c) \( \log_2 t + \log_2(t + 1) = 1 \)
\[ \log_2(t(t + 1)) = 1 \]
\[ 2^{\log_2(t(t + 1))} = 2^1 \]
\[ t(t + 1) = 2 \]
\[ t^2 + t - 2 = 0 \]
\[ (t + 2)(t - 1) = 0 \]
\[ t = -2, t = 1 \]

However, \( \log_2 t \) is undefined for negative \( t \), therefore we disregard the root \( t = -2 \).
Answer: \( t = 1 \)

(d) \( 10^{4x+1} = 300 \)
\[ \log_{10}(10^{4x+1}) = \log_{10} 300 \]
\[ 4x + 1 = \log_{10} 300 \]
\[ 4x = \log_{10} 300 - 1 \]
\[ x = \frac{\log_{10} 300 - 1}{4} \]

9. Evaluate the limits:

(a) \( \lim_{x \to 5} (7x - 25) = 7 \cdot 5 - 25 = 10 \)

(b) \( \lim_{x \to -1} \frac{x^3 + x^2}{x^2 + 3x + 2} = \lim_{x \to -1} \frac{x^2(x + 1)}{(x + 1)(x + 2)} = \lim_{x \to -1} \frac{x^2}{x + 2} = 1 \)

(c) \( \lim_{x \to 0} \frac{3 - \sqrt{9+x}}{x} = \lim_{x \to 0} \frac{(3 - \sqrt{9+x})(3 + \sqrt{9+x})}{x(3 + \sqrt{9+x})} = \lim_{x \to 0} 3^2 - (\sqrt{9+x})^2 = \lim_{x \to 0} \frac{9 - (9+x)}{x(3 + \sqrt{9+x})} = \lim_{x \to 0} \frac{-x}{x(3 + \sqrt{9+x})} = \lim_{x \to 0} \frac{-1}{3 + \sqrt{9} + x} = -\frac{1}{6} \)

(d) \( \lim_{x \to 2^+} \frac{x^3 - 2}{x^2 - x - 2} = \lim_{x \to 2^+} \frac{x^3 - 2}{(x - 2)(x + 1)} \quad \text{[pos.]} \quad \text{[small pos.](pos.)} = +\infty \)

(e) \( \lim_{x \to 2^-} \frac{x^3 - 2}{x^2 - x - 2} = \lim_{x \to 2^-} \frac{x^3 - 2}{(x - 2)(x + 1)} \quad \text{[pos.]} \quad \text{[small neg.](pos.)} = -\infty \)

(f) \( \lim_{x \to 2^-} \frac{x^3 - 2}{x^2 - x - 2} \) DNE because the limits in (d) and (e) are not equal
(g) \( \lim_{x \to 0} x^4 \cos \left( \frac{1}{x} \right) = 0 \) by the squeeze theorem since \(-x^4 \leq x^4 \cos \left( \frac{1}{x} \right) \leq x^4 \) and \( \lim_{x \to 0} (-x^4) = \lim_{x \to 0} (x^4) = 0 \).

(h) \( \lim_{x \to \infty} \frac{5x^3 - x - 3}{4x^3 + 3x^2 - 3} = \lim_{x \to \infty} \frac{5 - \frac{1}{x} - \frac{3}{x^2}}{4 + \frac{3}{x} - \frac{3}{x^2}} = \frac{5}{4} \)

(i) \( \lim_{x \to -\infty} \frac{5x^2 - x - 3}{4x^3 + 3x^2 - 3} = \lim_{x \to \infty} \frac{\frac{5}{x} - \frac{1}{x^2} - \frac{3}{x^3}}{4 + \frac{3}{x} - \frac{3}{x^2}} = 0 = 0 \)

(j) \( \lim_{x \to \infty} \frac{5x^3 - x - 3}{4x^2 + 3x - 3} = \lim_{x \to \infty} \frac{5x - \frac{1}{x} - \frac{3}{x^2}}{4 + \frac{3}{x} - \frac{3}{x^2}} = \infty \)

(k) \( \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{3x - 3} = \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{4 + \frac{5}{x^2}}}{3 - \frac{3}{x}} = \frac{2}{3} \)

(l) \( \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{3x - 3} = \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{4x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{4 + \frac{5}{x^2}}}{3 - \frac{3}{x}} = \frac{2}{3} \)

(m) \( \lim_{x \to \infty} (3 - x + 2x^2 - 5x^3) = \lim_{x \to \infty} x^3 \left( \frac{3}{x^3} - \frac{1}{x^2} + \frac{2}{x} - 5 \right) = -\infty \)

(n) \( \lim_{x \to -\infty} (3 - x + 2x^2 - 5x^3) = \lim_{x \to -\infty} x^3 \left( \frac{3}{x^3} - \frac{1}{x^2} + \frac{2}{x} - 5 \right) = \infty \)

10. Show that the equation \( x^5 - 4x + 2 = 0 \) has at least one solution in the interval (1, 2).

Let \( f(x) = x^5 - 4x + 2 \). Then \( f(x) \) is a continuous function with \( f(1) = -1 < 0 \) and \( f(2) = 26 > 0 \). By the intermediate value theorem, there is a point \( c \) between 1 and 2 such that \( f(c) = 0 \).

11. Find all values of \( c \) such that the function \( f(x) \) is continuous everywhere.

(a) \( f(x) = \begin{cases} \frac{cx}{5 - x} & \text{if } x \geq 2 \\ \frac{5}{x} & \text{if } x < 2 \end{cases} \)

Since linear functions are continuous everywhere, \( f(x) \) is continuous at all points except possibly at 2. It is continuous at 2 if and only if the functions \( \frac{cx}{5-x} \) and \( 5-x \) agree at 2 (that is, they have the same value at 2. The graph of \( f(x) \) then has no jump at 2.) So we set the values of \( \frac{cx}{5-x} \) and \( 5-x \) at 2 equal:
\( c \cdot 2 = 5 - 2 \)
\( 2c = 3 \)
\( c = \frac{3}{2} \)

(b) \( f(x) = \begin{cases} x^2 & \text{if } x \leq c \\ x^3 & \text{if } x > c \end{cases} \)

Since polynomial functions are continuous everywhere, \( f(x) \) is continuous at all points except possibly at \( c \). It is continuous at \( c \) if and only if the values of \( x^2 \) and \( x^3 \) agree at \( c \), i.e.
\( c^2 = c^3 \)
\( c^2 - c^3 = 0 \)
\( c^2(1 - c) = 0 \)
\( c = 0 \) or \( c = 1 \).
12. Find the following derivatives: $f(x) = \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)}$.

Since rational functions are continuous in their domains, $f(x)$ can have vertical asymptotes only at 5 and -7 (where it is undefined). Check the limits of $f(x)$ as $x$ approaches 5 and -7:

\[
\lim_{x \to 5^-} \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)} \left[ \frac{(\text{pos.})(\text{pos.})}{(\text{small pos.})(\text{pos.})} \right] = +\infty
\]
\[
\lim_{x \to 5^+} \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)} \left[ \frac{(\text{neg.})(\text{neg.})}{(\text{neg.})(\text{small pos.})} \right] = -\infty
\]

Since the limits are infinite, $f(x)$ has vertical asymptotes $x = 5$ and $x = -7$.

To find the horizontal asymptotes, we find the limits at infinity and negative infinity:

\[
\lim_{x \to \infty} \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)} = \lim_{x \to \infty} \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)} = \frac{3}{1} = 3
\]
\[
\lim_{x \to -\infty} \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)} = \lim_{x \to -\infty} \frac{(x + 2)(3x - 4)}{(x - 5)(x + 7)} = \frac{3}{1} = 3
\]

Thus $y = 3$ is the only horizontal asymptote.

13. Find the following derivatives:

(a) $f'(1)$ if $f(x) = 5$

$f'(1) = 0$ because the graph of $f(x)$ is a horizontal line, and its slope at 5 (as well as at any other point) is 0.

(b) $f'(2)$ if $f(x) = 7x - 3$

$f'(2) = 7$ because the graph of $f(x)$ is a line with slope 7. In particular, its slope at 2 is equal to 7.

(c) $f'(3)$ if $f(s) = s^2 + 5s - 6$

\[
f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3 + h)^2 + 5(3 + h) - 6 - 18}{h} = \frac{9 + 6h + h^2 + 15 + 5h - 24}{h} = \lim_{h \to 0} \frac{11h + h^2}{h} = \lim_{h \to 0} (11 + h) = 11
\]

(Equivalently, could use $f'(3) = \lim_{s \to 3} \frac{f(s) - f(3)}{s - 3}$.)

(d) $f'(4)$ if $f(t) = \sqrt{t}$

\[
f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4 + h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{(\sqrt{4 + h} - \sqrt{4})(\sqrt{4 + h} + \sqrt{4})}{h(\sqrt{4 + h} + \sqrt{4})} = \lim_{h \to 0} \frac{1}{\sqrt{4 + h} + \sqrt{4}} = \frac{1}{4}
\]

(Could use $f'(4) = \lim_{t \to 4} \frac{f(t) - f(4)}{t - 4}$.)

(e) $f'(5)$ if $f(x) = \frac{2}{x}$

\[
f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{2 - \frac{2}{x}}{x - 5} = \lim_{x \to 5} \frac{10 - 2x}{5x} = \lim_{x \to 5} \frac{10 - 2x}{5x} = \lim_{x \to 5} \frac{10 - 2x}{5x(x - 5)} = \lim_{x \to 5} \frac{2(5 - x)}{5x} = \lim_{x \to 5} \frac{-2}{5x} = -\frac{2}{25}
\]

(Could use $f'(5) = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h}$.)