Math 75A **Maria Nogin**

Review - 2

THEORY

Solving equations with logarithmic and exponential functions

Review laws of logarithms and laws of exponents (see Review-1).

Lines

The slope of the line that passes through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

An equation of the line that passes through the point $P(x_1, y_1)$ and has slope m is

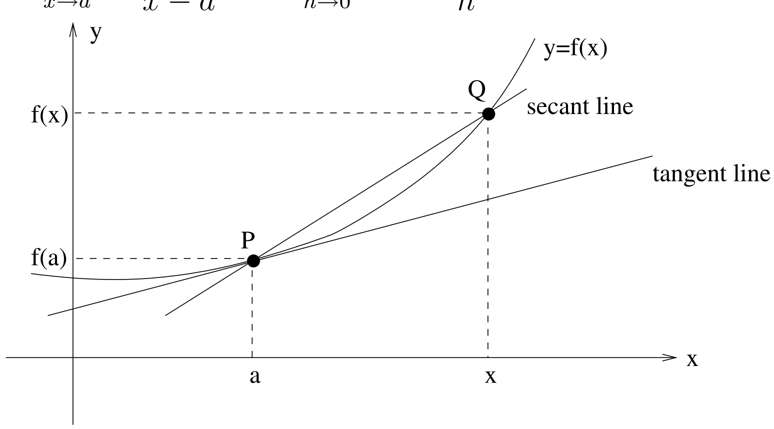
$$y - y_1 = m(x - x_1) \quad \text{(point-slope equation)}$$

To find an equation of the line that passes through points $P(x_1, y_1)$ and $Q(x_2, y_2)$, first find its slope and then use the point-slope equation.

Tangent line

A tangent line to a curve is a line that "touches" the curve. To find an equation of the tangent line to the curve $y = f(x)$ at the point P$(a, f(a))$, we have to know its slope. Let Q$(x, f(x))$ be another point on the curve. The slope of the secant line PQ is $m_{PQ} = \dfrac{f(x) - f(a)}{x - a}$. The slope of the tangent line at P is the limiting value of the slopes of secant lines PQ as Q approaches P, i.e.

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



An equation of the tangent line is $y - f(a) = m(x - a)$.

Useful formulas

$$(a+b)^2 = a^2 + 2ab + b^2, \qquad (a-b)^2 = a^2 - 2ab + b^2, \qquad (a+b)(a-b) = a^2 - b^2$$

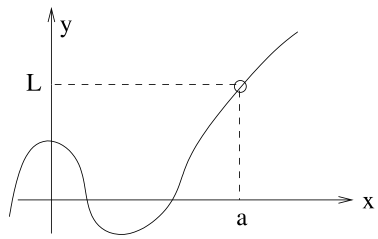
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

The limit of a function

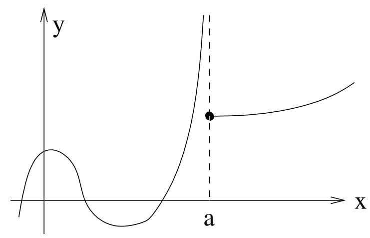
Def. We write $\lim\limits_{x \to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a, but not equal to a.

Note: The function $f(x)$ may or may not be defined at the point a.

If the values of $f(x)$ do not approach any number as x approaches a, we say that the limit $\lim\limits_{x \to a} f(x)$ does not exist.



$\lim\limits_{x \to a} f(x) = L$



$\lim\limits_{x \to a} f(x)$ DNE

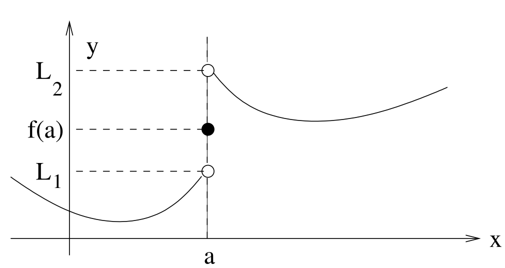
One-sided limits

Def. We write $\lim\limits_{x \to a^-} f(x) = L$ and say "the limit of $f(x)$ as x approaches a from the left is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Def. We write $\lim\limits_{x \to a^+} f(x) = L$ and say "the limit of $f(x)$ as x approaches a from the right is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a.

$$\lim_{x \to a} f(x) = L \;\text{ if and only if }\; \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

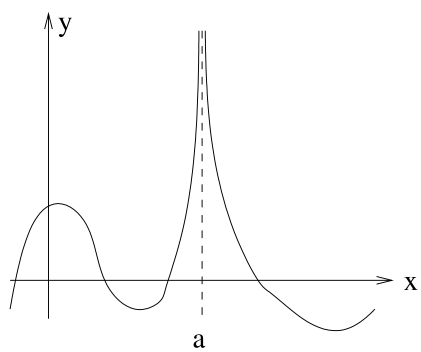
$\lim\limits_{x \to a^-} f(x) = L_1$
$\lim\limits_{x \to a^+} f(x) = L_2$
$\lim\limits_{x \to a} f(x)$ DNE b/c $L_1 \neq L_2$



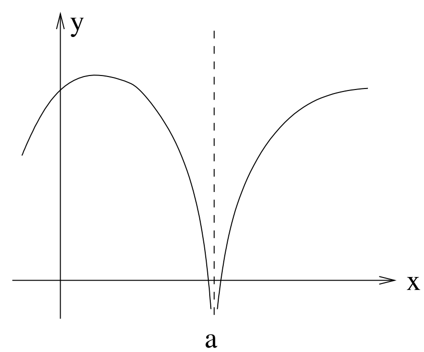
Infinite limits and vertical asymptotes

Def. $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

Def. $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be make arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

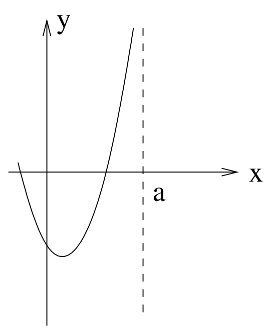


$\lim_{x \to a} f(x) = \infty$

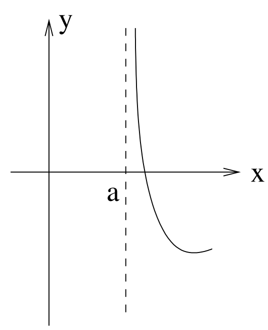


$\lim_{x \to a} f(x) = -\infty$

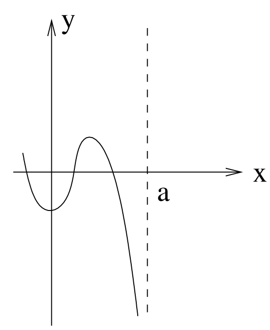
The definitions for one-sided infinite limits are similar. Here are the pictures:



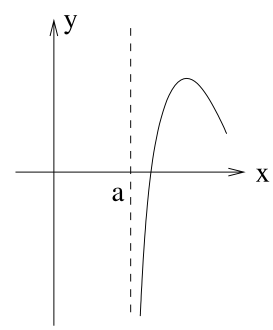
$\lim_{x \to a^-} f(x) = \infty$



$\lim_{x \to a^+} f(x) = \infty$



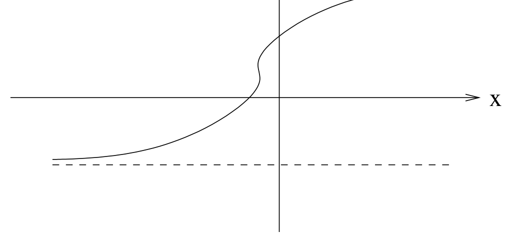
$\lim_{x \to a^-} f(x) = -\infty$



$\lim_{x \to a^+} f(x) = -\infty$

If at least one of these 4 conditions holds, the line $x = a$ is called a **vertical asymptote** of $y = f(x)$.

Limits at infinity and horizontal asymptotes



Def. $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large.

Def. $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large negative.

If at least one of the above conditions holds, then the line $y = L$ is called a **horizontal asymptote** of $y = f(x)$.

Thus, to find the horizontal asymptotes of $y = f(x)$, find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ (if they exist. If neither of these limits exist, then the curve $y = f(x)$ does not have horizontal asymptotes.)

Limit laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^c = [\lim_{x\to a} f(x)]^c$
7. $\lim_{x\to a} \sqrt[c]{f(x)} = \lim_{x\to a} \sqrt[c]{f(x)}$
8. $\lim_{x\to a} c = c$
9. $\lim_{x\to a} x = a$
10. $\lim_{x\to a} x^c = a^c$
11. $\lim_{x\to a} \sqrt[c]{x} = \sqrt[c]{a}$

If $P(x)$ is a polynomial function (i.e. $P(x) = a_n x^n + a_{n-1}x^{n-1} \dots a_1 x + a_0$), then for any c, $\lim_{x\to c} P(x) = P(c)$.

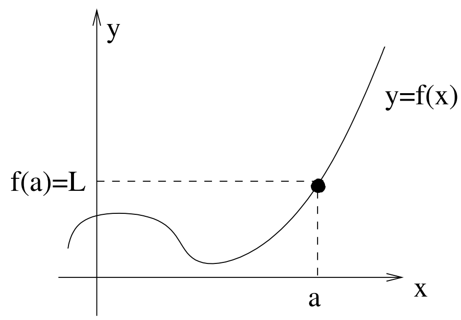
If $P(x)$ is a rational function (i.e. $P(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} \dots a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} \dots b_1 x + b_0}$), then for any c such that $P(x)$ is defined at c (i.e. the denominator is not equal to 0 at c), $\lim_{x\to c} P(x) = P(c)$.

Squeeze theorem. If $f(x) \leq g(x) \leq h(x)$ when x is near a, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Continuity

Def. A function $f(x)$ is continuous at a number a if

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$



If $f(x)$ is not continuous at a, we say f is discontinuous at a.

Def. A function $f(x)$ is continuous on an interval if it is continuous at every number in the interval.

Theorem. All power, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions, as well as all their combinations and compositions are continuous everywhere in their domain (i.e. wherever they are defined).

Intermediate Value Theorem. If $f(x)$ is continuous on $[a, b]$ and M is between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = M$.

Important Special Case. If $f(x)$ is continuous on $[a, b]$, and either $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$ (i.e. $f(x)$ changes sign on $[a, b]$), then there exists c in (a, b) such that $f(c) = 0$.

Derivative

Definition. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

(if this limit exists. In this case we say that $f(x)$ is differentiable at a.
If the limit does not exist, then $f(x)$ is not differentiable at a.)

$f'(a)$ is called the **rate of change** of $f(x)$ with respect to x at $(a, f(a))$.

The slope of the tangent line to $y = f(x)$ at $(a, f(a))$ is equal to $f'(a)$.

The tangent line to $y = f(x)$ at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.