Math 75A **Maria Nogin**

Review - 3

THEORY

Derivatives

Definition. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if this limit exists. In this case we say that $f(x)$ is differentiable at a. If the limit does not exist, then $f(x)$ is not differentiable at a.)

The derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

- Sum rule: $(f(x) + g(x))' = f'(x) + g'(x)$
- Difference rule: $(f(x) - g(x))' = f'(x) - g'(x)$
- Constant multiple rule: $(cf(x))' = cf'(x)$ for any constant c
- Product rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- Chain rule: $(f(g(x)))' = f'(g(x))g'(x)$

Derivatives of some important functions

- Constant function: $(c)' = 0$
- Power function: $(x^n)' = nx^{n-1}$
- Trigonometric functions: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$
  $(\tan x)' = \sec^2 x$, $(\cot x)' = -\csc x$, $(\sec x)' = \sec x \tan x$, $(\csc x)' = -\csc x \cot x$
- Exponential functions: $(e^x)' = e^x$, $(a^x)' = \ln(a)a^x$

Applications of derivatives

- The derivative $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at $(a, f(a))$. Thus the tangent line to $y = f(x)$ at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.
- The derivative $f'(a)$ is the rate of change of $y = f(x)$ when $x = a$.
  - If $s(t)$ is the position of an object at time t, then its velocity is $v(t) = s'(t)$, and its acceleration is $a(t) = v'(t)$.
  - If $n = f(t)$ is the size of a (plant or animal) population, then $f'(t)$ is the rate of growth of the population.
  - If $C(x)$ is the cost of producing x units of a certain product, then $C'(x)$ is the marginal cost function.

Implicit differentiation

is differentiation without solving for y. Be sure to use the chain rule when you differentiate a function of $y = y(x)$ with respect to x:

$$(f(y))' = (f(y(x)))' = f'(y(x)) \cdot y'(x)$$

Example.
If $\sqrt[3]{xy} = x^2y - 7x$ and $y(2) = 4$, find $y'(2)$.
Solution:
First rewrite the equation using $y(x)$ instead of just y:

$\sqrt[3]{xy(x)} = x^2y(x) - 7x$

$(xy(x))^{\frac{1}{3}} = x^2y(x) - 7x$

Now differentiate both sides of the equation with respect to x:

$$\frac{1}{3}(xy(x))^{-2/3}(1 \cdot y(x) + xy'(x)) = 2xy(x) + x^2y'(x) - 7$$

$$\frac{y(x)}{3(xy(x))^{2/3}} + \frac{xy'(x)}{3(xy(x))^{2/3}} = 2xy(x) + x^2y'(x) - 7$$

$$\frac{xy'(x)}{3(xy(x))^{2/3}} - x^2y'(x) = 2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}$$

$$y'(x)\left(\frac{x}{3(xy(x))^{2/3}} - x^2\right) = 2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}$$

$$y'(x) = \frac{2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}}{\left(\frac{x}{3(xy(x))^{2/3}} - x^2\right)}$$

if $x = 2$ and $y(2) = 4$, $\quad y'(2) = \dfrac{2 \cdot 2 \cdot 4 - 7 - \frac{4}{3(2\cdot4)^{2/3}}}{\left(\frac{2}{3(2\cdot4)^{2/3}} - 2^2\right)} = -\dfrac{52}{23}$

Trigonometry

$$180^{\text{O}} = \pi \text{ rad} \qquad 1^{\text{O}} = \frac{\pi}{180} \text{ rad} \qquad \left(\frac{180}{\pi}\right)^{\text{O}} = 1 \text{ rad}$$

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x$$

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x$$

Use the chart handed out in class to find the values of $\cos x$ and $\sin x$ at some important values of x.

Laws of exponents

- $a^b \cdot a^c = a^{b+c}$
- $\dfrac{a^b}{a^c} = a^{b-c}$
- $\left(a^b\right)^c = \left(a^c\right)^b = a^{bc}$
- $\sqrt[b]{a} = a^{1/b}$
- $\dfrac{1}{a^b} = a^{-b}$
- $a^0 = 1$
- $a^1 = a$

Fractions

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{ad}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \qquad \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}, \qquad \frac{a}{b} = \frac{ac}{bc}$$