Review - 1

THEORY

Useful formulas

$$(a+b)^2 = a^2 + 2ab + b^2, \qquad (a-b)^2 = a^2 - 2ab + b^2, \qquad (a+b)(a-b) = a^2 - b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Intervals

- $x \in (a, b) \iff a < x < b$
- $x \in [a, b) \iff a \leq x < b$
- $x \in (a, b] \iff a < x \leq b$
- $x \in [a, b] \iff a \leq x \leq b$
- $x \in (a, +\infty) \iff a < x$
- $x \in [a, +\infty) \iff a \leq x$
- $x \in (-\infty, b) \iff x < b$
- $x \in (-\infty, b] \iff x \leq b$

Absolute value

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Fractions

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{ad}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \qquad \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}, \qquad \frac{a}{b} = \frac{ac}{bc}$$

Distance formula

The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Lines

The slope of the line that passes through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of a horizontal line is equal to 0.
The slope of a vertical line is undefined.
An equation of the line that passes through the point $P(x_1, y_1)$ and has slope m is

$$y - y_1 = m(x - x_1) \quad \text{(point-slope equation)}$$

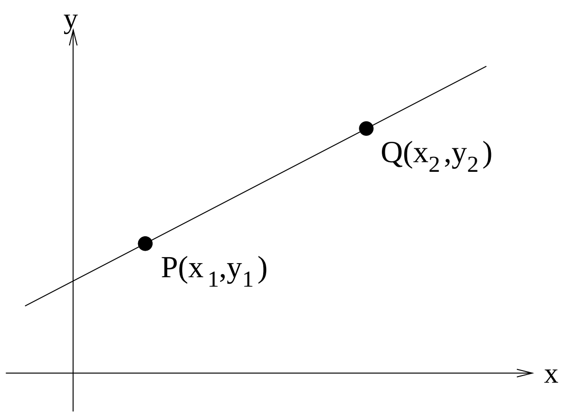
To find an equation of the line that passes through points $P(x_1, y_1)$ and $Q(x_2, y_2)$, first find its slope and then use the point-slope equation.

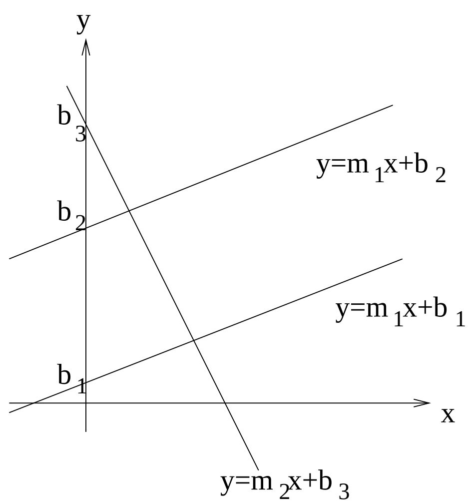
An equation of the line that has slope m and intersects the y-axis at the point $(0, b)$ is

$$y = mx + b \quad \text{(slope-intercept equation)}$$

Two non-vertical lines are parallel if and only if they have the same slope.
Two non-vertical lines with slopes m_1 and m_2 are perpendicular if and only if

$$m_1 \cdot m_2 = -1.$$





Circles

An equation of a circle with center at (a, b) and radius r can be written in the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

Domain and range of a function

The **domain** of $f(x)$ is the set of all values of x for which $f(x)$ is defined.
The **range** of $f(x)$ is the set of all values of $y = f(x)$.

Important classes of functions

- Constant: $f(x) = a$
- Linear: $f(x) = mx + b$
- Quadratic: $f(x) = ax^2 + bx + c$
- Cubic: $f(x) = ax^3 + bx^2 + cx + d$
- Polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
- Rational: $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials
- Power: $f(x) = x^n$
- Root: $f(x) = \sqrt[n]{x}$
- Trigonometric: $f(x) = \sin x, \cos x, \tan x, \ldots$
- Exponential: $f(x) = a^x$
- Logarithmic: $f(x) = \log_a x$

See section 1.2 for **GRAPHS** of the above classes of functions.

Trigonometry

$$180^{\mathrm{O}} = \pi \text{ rad} \qquad 1^{\mathrm{O}} = \frac{\pi}{180} \text{ rad}$$

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$$

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x, \quad \tan(x + \pi) = \tan x$$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0

Combinations of functions

$$(f \pm g)(x) = f(x) \pm g(x), \qquad (fg)(x) = f(x)g(x), \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

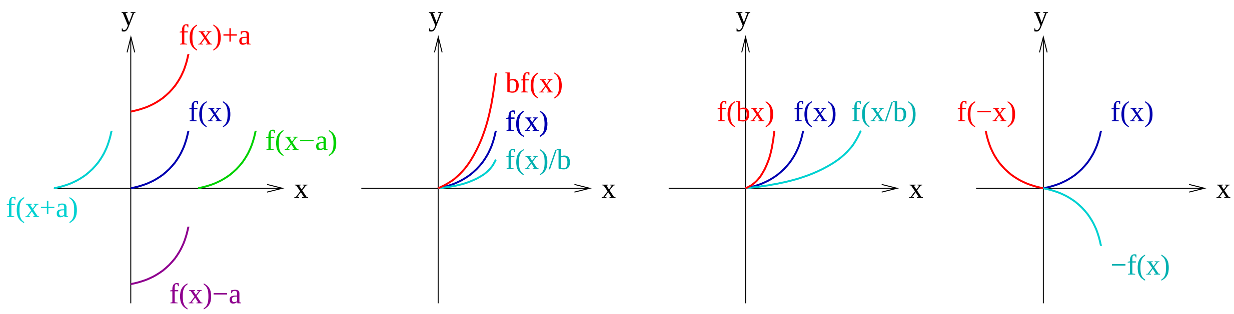
Composition of functions

$$(f \circ g)(x) = f(g(x))$$

Transformations of functions

Let $a > 0$ and $b > 1$. To obtain the graph of

- $y = f(x) + a$, shift the graph of $y = f(x)$ a units upward.
- $y = f(x) - a$, shift the graph of $y = f(x)$ a units downward.
- $y = f(x + a)$, shift the graph of $y = f(x)$ a units to the left.
- $y = f(x - a)$, shift the graph of $y = f(x)$ a units to the right.
- $y = bf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of b.
- $y = \dfrac{f(x)}{b}$, compress the graph of $y = f(x)$ vertically by a factor of b.
- $y = f(bx)$, compress the graph of $y = f(x)$ horizontally by a factor of b.
- $y = f\left(\dfrac{x}{b}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of b.
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

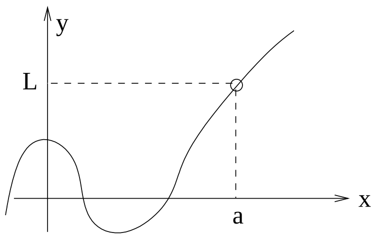


The limit of a function

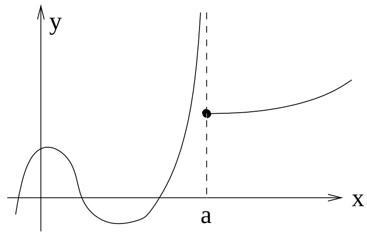
Def. We write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficienly close to a, but not equal to a.

Note: The function $f(x)$ may or may not be defined at the point a.

If the values of $f(x)$ do not approach any number as x approaches a, we say that the limit $\lim_{x\to a} f(x)$ does not exist.



$$\lim_{x\to a} f(x) = L$$



$$\lim_{x\to a} f(x) \text{ DNE}$$

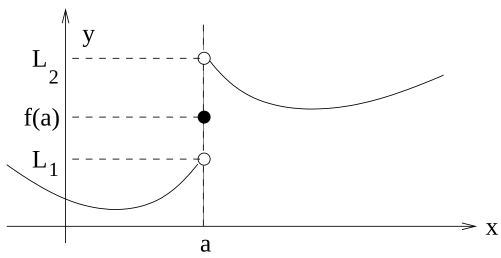
One-sided limits

Def. We write $\lim_{x\to a^-} f(x) = L$ and say "the limit of $f(x)$ as x approaches a from the left is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Def. We write $\lim_{x\to a^+} f(x) = L$ and say "the limit of $f(x)$ as x approaches a from the right is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a.

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

$\lim_{x\to a^-} f(x) = L_1$
$\lim_{x\to a^+} f(x) = L_2$
$\lim_{x\to a} f(x)$ DNE b/c $L_1 \neq L_2$



Limit laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^c = [\lim_{x\to a} f(x)]^c$
7. $\lim_{x\to a} \sqrt[c]{f(x)} = \lim_{x\to a} \sqrt[c]{f(x)}$
8. $\lim_{x\to a} c = c$
9. $\lim_{x\to a} x = a$
10. $\lim_{x\to a} x^c = a^c$
11. $\lim_{x\to a} \sqrt[c]{x} = \sqrt[c]{a}$

If $P(x)$ is a polynomial function (i.e. $P(x) = a_n x^n + a_{n-1}x^{n-1} \ldots a_1 x + a_0$), then for any c, $\lim_{x\to c} P(x) = P(c)$.

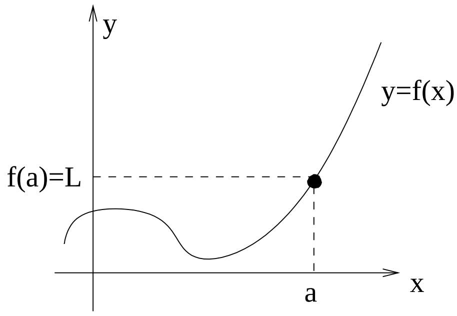
If $P(x)$ is a rational function (i.e. $P(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} \ldots a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} \ldots b_1 x + b_0}$), then for any c such that $P(x)$ is defined at c (i.e. the denominator is not equal to 0 at c), $\lim_{x\to c} P(x) = P(c)$.

Squeeze theorem. If $f(x) \leq g(x) \leq h(x)$ when x is near a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Continuity

Def. A function $f(x)$ is continuous at a number a if

1. $f(a)$ is defined.
2. $\lim\limits_{x\to a} f(x)$ exists.
3. $\lim\limits_{x\to a} f(x) = f(a)$



If $f(x)$ is not continuous at a, we say f is discontinuous at a.

Def. A function $f(x)$ is continuous on an interval if it is continuous at every number in the interval.

Theorem. All power, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions, as well as all their combinations and compositions are continuous everywhere in their domain (i.e. wherever they are defined).

Intermediate Value Theorem. If $f(x)$ is continuous on $[a, b]$ and M is between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = M$.

Important Special Case. If $f(x)$ is continuous on $[a, b]$, and either $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$ (i.e. $f(x)$ changes sign on $[a, b]$), then there exists c in (a, b) such that $f(c) = 0$.

Other properties of functions

A function $f(x)$ is called **even** if $f(-x) = f(x)$ for all x in the domain of f. The graph of an even function is symmetric about the y-axis.

A function $f(x)$ is called **odd** if $f(-x) = -f(x)$ for all x in the domain of f. The graph of an odd function is symmetric about the origin.

A function $f(x)$ is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

A function $f(x)$ is called **deccreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.