Math 75A **Maria Nogin**

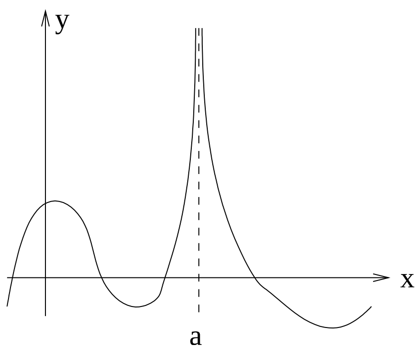
Review - 2

THEORY

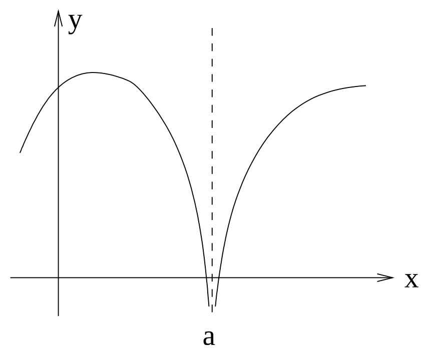
Infinite limits and vertical asymptotes

Def. $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

Def. $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

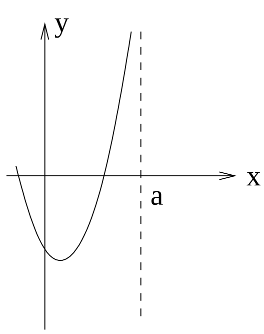


$\lim_{x \to a} f(x) = \infty$

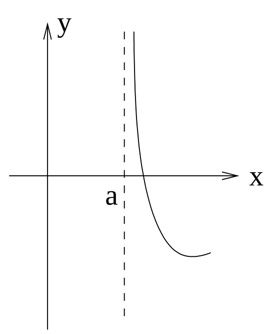


$\lim_{x \to a} f(x) = -\infty$

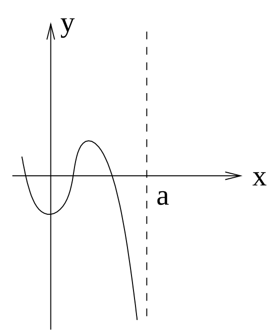
The definitions for one-sided infinite limits are similar. Here are the pictures:



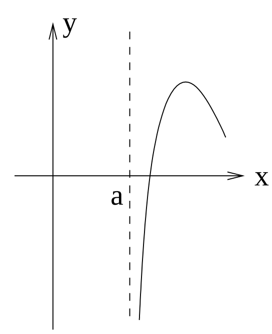
$\lim_{x \to a^-} f(x) = \infty$



$\lim_{x \to a^+} f(x) = \infty$



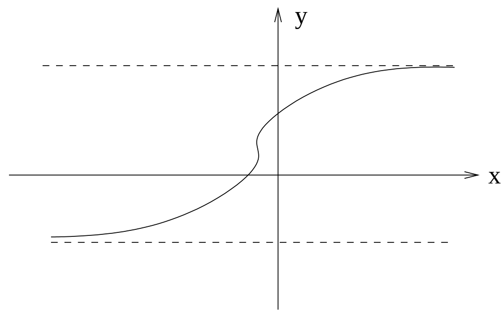
$\lim_{x \to a^-} f(x) = -\infty$



$\lim_{x \to a^+} f(x) = -\infty$

If at least one of these 4 conditions holds, the line $x = a$ is called a **vertical asymptote** of $y = f(x)$.

Limits at infinity and horizontal asymptotes



Def. $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large.

Def. $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large negative.

If at least one of the above conditions holds, then the line $y = L$ is called a **horizontal asymptote** of $y = f(x)$.

Thus, to find the horizontal asymptotes of $y = f(x)$, find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ (if they exist. If neither of these limits exist, then the curve $y = f(x)$ does not have horizontal asymptotes.)

Useful formulas

$$(a+b)^2 = a^2 + 2ab + b^2, \qquad (a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2), \qquad a^3 + b^3 = (a-b)(a^2 + ab + b^2)$$

Derivative

Definition. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if this limit exists. In this case we say that $f(x)$ is differentiable at a.
If the limit does not exist, then $f(x)$ is not differentiable at a.)

$f'(a)$ is called the **rate of change** of $f(x)$ with respect to x at $(a, f(a))$.

The slope of the tangent line to $y = f(x)$ at $(a, f(a))$ is equal to $f'(a)$.

The tangent line to $y = f(x)$ at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.

The derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

- Sum rule: $(f(x) + g(x))' = f'(x) + g'(x)$
- Difference rule: $(f(x) - g(x))' = f'(x) - g'(x)$
- Constant multiple rule: $(cf(x))' = cf'(x)$ for any constant c
- Product rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- Chain rule: $(f(g(x)))' = f'(g(x))g'(x)$

Derivatives of some important functions

- Constant function: $(c)' = 0$
- Power function: $(x^n)' = nx^{n-1}$
- Special cases of the power function: $(x)' = 1$, $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, $\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$
- Trigonometric functions: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

  $(\tan x)' = \sec^2 x$, $(\cot x)' = -\csc^2 x$,

  $(\sec x)' = \sec x \tan x$, $(\csc x)' = -\csc x \cot x$

Applications of derivatives

- The derivative $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at $(a, f(a))$. Thus the tangent line to $y = f(x)$ at $(a, f(a))$ has equation

$$y - f(a) = f'(a)(x - a).$$

- The derivative $f'(a)$ is the rate of change of $y = f(x)$ when $x = a$.
  - If $s(t)$ is the position of an object at time t, then its velocity is $v(t) = s'(t)$, and its acceleration is $a(t) = v'(t)$.
  - If $n = f(t)$ is the size of a (plant or animal) population, then $f'(t)$ is the rate of growth of the population.
  - If $C(x)$ is the cost of producing x units of a certain product, then $C'(x)$ is the marginal cost function.

Laws of exponents

- $(ab)^c = a^c b^c$
- $a^b \cdot a^c = a^{b+c}$
- $\dfrac{a^b}{a^c} = a^{b-c}$
- $\left(a^b\right)^c = \left(a^c\right)^b = a^{bc}$
- $\sqrt[b]{a} = a^{1/b}$
- $\dfrac{1}{a^b} = a^{-b}$
- $a^0 = 1$
- $a^1 = a$