Math 75B **Maria Nogin**

Practice test 2 - Solutions

Multiple choice questions: circle the correct answer

1. If a bacteria population has initial size of 100 and doubles every hour, what is, approximately, the size of the population after 2.5 hours?

   A. 255 **B.** 300 **C.** 450 **D.** 566 **E.** 600

   (The population function is $P(t) = 100e^{kt}$. At $t = 1$ we have $200 = 100e^k$, so $e^k = 2$. Then $P(2.5) = 100e^{k \cdot 2.5} = 100(e^k)^{2.5} = 100 \cdot 2^{2.5} = 100 \cdot 2^2\sqrt{2} = 400\sqrt{2} \approx 566$.)

2. Find the exact value of $\arccos\left(-\frac{1}{2}\right)$.

   A. $-\frac{\pi}{3}$ **B.** $-\frac{\pi}{6}$ **C.** $\frac{\pi}{6}$ **D.** $\frac{\pi}{3}$ **E.** $\frac{2\pi}{3}$

   (By definition, because $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $0 \le \frac{2\pi}{3} \le \pi$.)

3. Find the derivative of $\arctan x + \arctan 2$.

   A. $\frac{1}{1+x^2}$ **B.** $\frac{1}{6+x^2}$ **C.** $\frac{1}{1+x^2} + \frac{1}{5}$ **D.** $\frac{1}{(1+x^2)5}$
   E. none of the above

   (Because $(\arctan x)' = \frac{1}{1+x^2}$ and $\arctan 2$ is a constant, so its derivative is 0.)

4. If $f(1) = 3$ and $f'(x) \le 2$, which of the following must be true about $f(4)$?

   A. $f(4) \le 6$ **B.** $f(4) \le 9$ **C.** $f(4) \le 11$ **D.** $f(4) \ge 5$ **E.** $f(4) \ge 8$

   (By the Mean Value Theorem, there is a point c such that $f'(c) = \dfrac{f(4) - f(1)}{4-1}$. Since $f'(x) \le 2$, we have $\dfrac{f(4) - f(1)}{4-1} \le 2$. Then $f(4) - f(1) \le 6$, so $f(4) \le f(1) + 6 = 9$.)

5. Use Newton's method to approximate the root of the equation $x^2 - 23 = 0$. Let $x_1 = 5$. Find x_2.

   A. 0 **B.** 0.3 **C.** 4.8 **D.** 5 **E.** 5.2

   (Let $f(x) = x^2 - 23$. Then $f'(x) = 2x$, and we have $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5 - \frac{f(5)}{f'(5)} = 5 - \frac{2}{10} = 4.8$.)

6. Which of the following functions is an antiderivative of $\sqrt{x}$?

   A. $2\sqrt{x}$ **B.** $\frac{1}{2\sqrt{x}}$ **C.** $\frac{3}{2}x^{\frac{3}{2}}$ **D.** $\frac{2x\sqrt{x}}{3}$
   E. none of the above

   (An antiderivative of $x^{1/2}$ is $\frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3} = \frac{2x\sqrt{x}}{3}$.)

Regular problems: show all your work

7. Using implicit differentiation we have:

   $2xy + x^2y' + 3y + 3xy' + 4y^2 + 8xyy' = 0$

   $(x^2 + 3x + 8xy)y' = -2xy - 3y - 4y^2$

   $y' = \dfrac{-2xy - 3y - 4y^2}{x^2 + 3x + 8xy}$

   If $x = 2$ and $y = 3$, $y' = \frac{-12-9-36}{4+6+48} = -\frac{57}{58}$.

   So the slope of the tangent line is $-\frac{57}{58}$.

   An equation is then $y - 3 = -\frac{57}{58}(x - 2)$

   $y - 3 = -\frac{57}{58}x + \frac{57}{29}$

   $y = -\frac{57}{58}x + \frac{57}{29} + 3$

   $y = -\frac{57}{58}x + \frac{144}{29}$

8. Let a and b be the legs and let c be the hypothenuse. Then $a^2 + b^2 = c^2$. Differentiating (implicitly, with respect to time t) gives

   $2aa' + 2bb' = 2cc'$, so $aa' + bb' = cc'$.

   We are given $a' = 4$ and $b' = 4$. If $a = 5$ and $b = 12$, by Pythagorean theorem $c = \sqrt{5^2 + 12^2} = 13$.

   So we have $5 \cdot 4 + 12 \cdot 4 = 13c'$

   $68 = 13c'$

   $c' = \frac{68}{13}$, i.e. the hypothenuse is increasing at a rate of $\frac{68}{13}$ cm/min.

9. We will use the closed interval method.

   First find the critical numbers: $f'(x) = 1 - \frac{8}{x^3} = 0$

   $1 = \frac{8}{x^3}$

   $x^3 = 8$

   $x = 2$.

   The value of the function at the critical number is $f(2) = 2 + \frac{4}{4} = 3$.

   Next find the value of the function at each endpoint: $f(1) = 1 + \frac{4}{1} = 5$, $f(4) = 4 + \frac{4}{16} = 4.25$.

   The largest of this values, i.e. 5, is the absolute maximum value, and the smallest, i.e. 3, is the absolute minimum value of the function on the given interval.

10. Using L'Hospital's rule:

    (a) $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

    (b) $\lim_{x\to 0} \frac{\sin(2x)}{\tan(3x)} = \frac{2\cos(2x)}{3\sec^2(3x)} = \frac{2}{3}$

11. $f(x) = x^2 e^x$.

    (a) Domain: $(-\infty, +\infty)$

    (b) $f(0) = 0$; the only solution of $x^2e^x = 0$ is $x = 0$, so the only intercept is $(0, 0)$.

    (c) There are no vertical asymptotes since the function is continuous everywhere.
    Horizontal asymptotes: $\lim_{x\to+\infty} x^2e^x = +\infty$;
    $\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = \lim_{x\to-\infty} 2e^x = 0$, so $y = 0$ is a horizontal asymptote.

    (d) Critical numbers: $f'(x) = 2xe^x + x^2e^x = 0$
    $xe^x(2 + x) = 0$
    $x = 0$, $x = -2$.

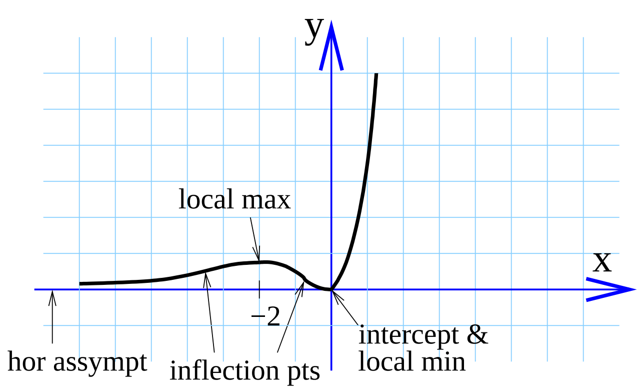
    (e) Since $f'(-3) > 0$, $f'(-1) < 0$, and $f'(1) > 0$, the function is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, and increasing on $(0, +\infty)$.

    (f) $x = -2$ is a local maximum and $x = 0$ is a local minimum.

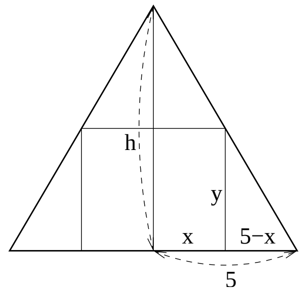
    (g) $f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2)$
    $e^x(2 + 4x + x^2) = 0$
    $2 + 4x + x^2 = 0$
    $x = -2 \pm \sqrt{2}$
    Since $f''(-4) > 0$, $f''(-2) < 0$, and $f''(0) > 0$, the function is concave upward on $(-\infty, -2-\sqrt{2})$, concave downward on $(-2-\sqrt{2}, -2+\sqrt{2})$, and concave upward on $(-2+\sqrt{2}, +\infty)$.

    (h) Inflection points: $x = -2 \pm \sqrt{2}$

    (i) Graph of $f(x)$:



12. We want to find x and y such that the area of the rectangle, i.e. $2xy$, is a maximum.



Use similar triangles to find a relationship between x and y, e.g.

$\frac{5-x}{5} = \frac{y}{h}$ where $h = 5\tan 60^0 = 5\sqrt{3}$, so we have

$\frac{5-x}{5} = \frac{y}{5\sqrt{3}}$

Multiplying both sides by $5\sqrt{3}$ gives $\sqrt{3}(5-x) = y$.

Now we can express the area as a function of one variable x:

$A(x) = 2x\sqrt{3}(5-x) = 10\sqrt{3}x - 2\sqrt{3}x^2$. To find a maximum, we have to differentiate $A(x)$ and set the derivative equal to 0:

$A'(x) = 10\sqrt{3} - 4\sqrt{3}x = 0$

$10\sqrt{3} = 4\sqrt{3}x$

$x = 2.5$

Since $A'(x)$ changes from positive to negative at 2.5, this is a local maximum.

$y = \sqrt{3}(5-x) = 2.5\sqrt{3}$, thus the width of the rectangle is $2x = 2 \cdot 2.5 = 5$, and the height is $y = 2.5\sqrt{3}$.

13. (a) $f(x) = x - 2x^4 - 2\cos x - \sin x + c$
    $f(0) = -2 + c = 5$
    so $c = 7$
    $f(x) = x - 2x^4 - 2\cos x - \sin x + 7$

    (b) $f'(x) = 6x - 8x^3 + c$
    $f'(1) = 6 - 8 + c = -3$
    so $c = -1$
    $f'(x) = 6x - 8x^3 - 1$
    $f(x) = 3x^2 - 2x^4 - x + d$
    $f(2) = 12 - 32 - 2 + d = -32$
    so $d = -10$
    $f(x) = 3x^2 - 2x^4 - x - 10$