Math 75B **Maria Nogin**

Practice test 2 - Solutions

Multiple choice questions: circle the correct answer

1. If $f(1) = 3$ and $f'(x) \leq 2$, which of the following must be true about $f(4)$?

   A. $f(4) \leq 6$ (**B.**) $f(4) \leq 9$ **C.** $f(4) \leq 11$ **D.** $f(4) \geq 5$ **E.** $f(4) \geq 8$

   (By the Mean Value Theorem, there is a point c such that $f'(c) = \dfrac{f(4) - f(1)}{4 - 1}$. Since $f'(x) \leq 2$, we have $\dfrac{f(4) - f(1)}{4 - 1} \leq 2$. Then $f(4) - f(1) \leq 6$, so $f(4) \leq f(1) + 6 = 9$.)

2. Use Newton's method to approximate the root of the equation $x^2 - 23 = 0$. Let $x_1 = 5$. Find x_2.

   A. 0 **B.** 0.3 (**C.**) 4.8 **D.** 5 **E.** 5.2

   (Let $f(x) = x^2 - 23$. Then $f'(x) = 2x$, and we have $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5 - \frac{f(5)}{f'(5)} = 5 - \frac{2}{10} = 4.8$.)

3. Which of the following functions is an antiderivative of $\sqrt{x}$?

   A. $2\sqrt{x}$ **B.** $\dfrac{1}{2\sqrt{x}}$ **C.** $\dfrac{3}{2}x^{\frac{3}{2}}$ (**D.**) $\dfrac{2x\sqrt{x}}{3}$
   E. none of the above

   (An antiderivative of $x^{1/2}$ is $\frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3} = \frac{2x\sqrt{x}}{3}$.)

4. Which of the following is true about the function $f(x) = \ln x$?

   A. its domain is $(-\infty, \infty)$ (**B.**) it is increasing everywhere in its domain
   C. it is concave up everywhere in its domain **D.** it has one critical number
   E. none of the above

   (Review the graph.)

5. Which of the following functions has exactly one inflection point?

   A. $f(x) = \frac{1}{x}$ **B.** $g(x) = e^x$ **C.** $h(x) = x^2$ (**D.**) $j(x) = x^3$ **E.** $k(x) = \sin(x)$

   (The first three functions do not have any inflection points. The last one has infinitely many.)

6. Evaluate the limit: $\displaystyle\lim_{x \to 0} \frac{9^x - 3^x}{4^x - 2^x}$

   A. 3 (**B.**) $\frac{\ln 3}{\ln 2}$ **C.** 4 **D.** $\frac{\ln 6}{\ln 2}$
   E. none of the above

(Use L'Hospital's Rule and then simplify: $\lim_{x\to 0}\dfrac{9^x-3^x}{4^x-2^x} = \lim_{x\to 0}\dfrac{\ln 9\cdot 9^x-\ln 3\cdot 3^x}{\ln 4\cdot 4^x-\ln 2\cdot 2^x} = \dfrac{\ln 9-\ln 3}{\ln 4-\ln 2} = \dfrac{2\ln 3-\ln 3}{2\ln 2-\ln 2} = \dfrac{\ln 3}{\ln 2}$.)

Regular problems: show all your work

7. $f'(x) = 3x^2 - 6x = 0$ gives $x = 0$ and $x = 2$. Since $f'(-1) > 0$, $f'(1) < 0$, and $f'(3) > 0$, the point $x = 0$ is a local maximum and the point $x = 2$ is a local minimum.

8. Use the closed interval method:
1. Find the critical numbers: $f'(x) = 4x^3 + 12x^2 = 0$ gives $x = 0$ and $x = -3$. However, -3 is not in our interval. Only 0 is.
2. Find the value of the function at the critical number(s): $f(0) = 5$.
3. Find the value of the function at the endpoints of the interval: $f(-2) = -11$. The other endpoint is 0, but we already found the value at 0.
4. The largest of the above values, i.e. 5, is the absolute maximum value of the function on the given interval, and the smallest of the above values, i.e. -11, is the absolute minimum value.

9. Use the closed interval method:
1. Find the critical numbers: $f'(x) = \cos x$. The only root on the given interval is $x = \frac{\pi}{2}$.
2. Find the value of the function at the critical number(s): $f\left(\frac{\pi}{2}\right) = 1$.
3. Find the value of the function at the endpoints of the interval: $f(0) = 0$, $f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.
4. The largest of the above values, i.e. 1, is the absolute maximum value of the function on the given interval, and the smallest of the above values, i.e. $-\frac{1}{\sqrt{2}}$, is the absolute minimum value.

10. Let $f(x) = x^7 + 3x^3 + x$. Since the funtion $f(x)$ is continuous, $f(0) < 4$ and $f(1) > 4$, by the Intermediate Value Theorem the equation $f(x) = 4$ has at least one real root. However, since $f'(x) = 7x^6 + 9x^2 + 1 > 0$, by Rolle's Theorem this equation cannot have more than one real root. Thus it has exactly one real root.

11. $f(x) = x^2 e^x$.

    (a) Domain: $(-\infty, +\infty)$

    (b) $f(0) = 0$; the only solution of $x^2e^x = 0$ is $x = 0$, so the only intercept is $(0, 0)$.

    (c) There are no vertical asymptotes since the function is continuous everywhere. Horizontal asymptotes: $\lim_{x\to+\infty} x^2e^x = +\infty$;
    $\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty}\dfrac{x^2}{e^{-x}} = \lim_{x\to-\infty}\dfrac{2x}{-e^{-x}} = \lim_{x\to-\infty}\dfrac{2}{e^{-x}} = \lim_{x\to-\infty} 2e^x = 0$, so $y = 0$ is a horizontal asymptote.

(d) Critical numbers: $f'(x) = 2xe^x + x^2e^x = 0$
$xe^x(2+x) = 0$
$x = 0$, $x = -2$.

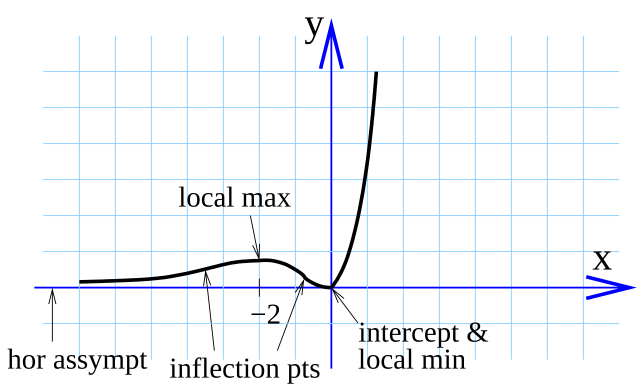
(e) Since $f'(-3) > 0$, $f'(-1) < 0$, and $f'(1) > 0$, the function is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, and increasing on $(0, +\infty)$.

(f) $x = -2$ is a local maximum and $x = 0$ is a local minimum.

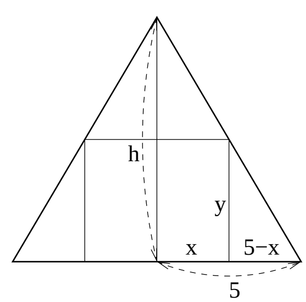
(g) $f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2)$
$e^x(2 + 4x + x^2) = 0$
$2 + 4x + x^2 = 0$
$x = -2 \pm \sqrt{2}$
Since $f''(-4) > 0$, $f''(-2) < 0$, and $f''(0) > 0$, the function is concave upward on $(-\infty, -2-\sqrt{2})$, concave downward on $(-2-\sqrt{2}, -2+\sqrt{2})$, and concave upward on $(-2+\sqrt{2}, +\infty)$.

(h) Inflection points: $x = -2 \pm \sqrt{2}$

(i) Graph of $f(x)$:



12. We want to find x and y such that the area of the rectangle, i.e. $2xy$, is a maximum.



Use similar triangles to find a relationship between x and y, e.g.

$\frac{5-x}{5} = \frac{y}{h}$ where $h = 5\tan 60^0 = 5\sqrt{3}$, so we have

$\frac{5-x}{5} = \frac{y}{5\sqrt{3}}$

Multiplying both sides by $5\sqrt{3}$ gives $\sqrt{3}(5-x) = y$.

Now we can express the area as a function of one variable x:

$A(x) = 2x\sqrt{3}(5-x) = 10\sqrt{3}x - 2\sqrt{3}x^2$. To find a maximum, we have to differentiate $A(x)$ and set the derivative equal to 0:

$A'(x) = 10\sqrt{3} - 4\sqrt{3}x = 0$

$10\sqrt{3} = 4\sqrt{3}x$

$x = 2.5$

Since $A'(x)$ changes from positive to negative at 2.5, this is a local maximum.

$y = \sqrt{3}(5-x) = 2.5\sqrt{3}$, thus the width of the rectangle is $2x = 2 \cdot 2.5 = 5$, and the height is $y = 2.5\sqrt{3}$.

13. $f(x) = x - 2x^4 - 2\cos x - \sin x + c$
    $f(0) = -2 + c = 5$
    $so\ \ c = 7$
    $f(x) = x - 2x^4 - 2\cos x - \sin x + 7$