MATH 75B

Test 1 - Solutions

1. C.
   Since the size of the population at $t = 0$ is $P(0) = Ce^{k\cdot 0} = Ce^0 = C$.

2. E.
   Using the definition, since $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $0 \le \frac{5\pi}{6} \le \pi$.

3. C.
   $sin(\pi) = 0$; $arcsin(0) = 0$ using the definition (since $\sin(0) = 0$ and $-\frac{\pi}{2} \le 0 \le \frac{\pi}{2}$).

4. B.
   Using L'Hospital's Rule: $\displaystyle\lim_{x\to\infty} \frac{3x}{e^x} = \lim_{x\to\infty} \frac{3}{e^x} = 0$.

5. E.
   $f'(x) = 3x^2 - 12 = 0$ gives $3x^2 = 12$; $x^2 = 4$; $x = \pm 2$.

6. C.
   Namely, let $f(x) = x^3 + x + 1$. If the equation had at least two roots, then by the Mean Value Theorem there must be a point at which $f'(x) = 0$. However, $f'(x) = 3x^2 + 1 > 0$ for all x, so the above is not possible.

7. (a) $2xy + x^2y' + y^3 + 3xy^2y' = 4$
   $x^2y' + 3xy^2y' = 4 - 2xy - y^3$
   $(x^2 + 3xy^2)y' = 4 - 2xy - y^3$
   $$y'(x) = \frac{4 - 2xy - y^3}{x^2 + 3xy^2}$$
   (b) $(-1)^2 \cdot 2 + (-1) \cdot 2^3 = 4 \cdot (-1) - 2$
   (c) $y'(-1) = \frac{4-2(-1)2-2^3}{(-1)^2+3(-1)2^2} = 0$.

8. Let C be the point from which ship A started and let D be the point from which ship B started. Then $|CD| = 100$. Let A be the current position of ship A and let B the current position of ship B. Let $x = |AD|$, $y = |BD|$, and $z = |AB|$. By Pythagorean theorem, $x^2 + y^2 = z^2$. Differentiating both sides of this equation implicitly with respect to t (time) gives $2xx' + 2yy' = 2zz'$, thus $xx' + yy' = zz'$. At 3PM, $x = 100 - 3 \cdot 30 = 10$ km, $y = 3 \cdot 20 = 60$ km and z can be found using the Pythagorean theorem again: $10^2 + 60^2 = z^2$, so $z = \sqrt{3700} = 10\sqrt{37}$.
   Next, $x' = -30$ and $y' = 20$. Plugging all the known values into $xx' + yy' = zz'$ gives
   $10(-30) + 60 \cdot 20 = 10\sqrt{37}z'$
   $-300 + 1200 = 10\sqrt{37}z'$

$900 = 10\sqrt{37}z'$
$z' = 90/\sqrt{37}$ km/h

9. The mass is given by $m(t) = Ce^{kt}$ where C is the initial amount. We are told that $m(1) = .945C$, so $Ce^k = .945C$ (where time is measured in years). Therefore $e^k = .945$. We have to find the half-life, i.e. the time t such that $m(t) = \frac{1}{2}C$. Thus
$\frac{1}{2}C = Ce^{kt}$
$\frac{1}{2} = (e^k)^t$
$\frac{1}{2} = .945^t$
$t = \log_{.945}(1/2)$

10. (a) $f'(x) = \dfrac{\arcsin(x)}{2\sqrt{x}} + \dfrac{\sqrt{x}}{\sqrt{1-x^2}}$ (using the product rule)

 (b) $g'(x) = -\dfrac{1}{(1+\left(\frac{1}{x}\right)^2)x^2} = -\dfrac{1}{(x+1)^2)}$ (using the chain rule)

11. Rewriting the expression as a fraction and then using L'Hospital's rule, simplifying, and using L'Hospital's rule again:
$\lim_{x\to1}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right) = \lim_{x\to1}\dfrac{x-1-\ln x}{(\ln x)(x-1)} = \lim_{x\to1}\dfrac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} =$
$\lim_{x\to1}\dfrac{x-1}{(x-1)+x\ln x} = \lim_{x\to1}\dfrac{1}{1+\ln x+1} = 0.5$

12. Using the closed interval method:
$f'(x) = \dfrac{x^2+9-2x^2}{(x^2+9)^2} = \dfrac{9-x^2}{(x^2+9)^2}$. The derivative is 0 when $9 - x^2 = 0$, i.e. $x = \pm3$. However, the value -3 is outside of the given interval. Since $f(x)$ is a rational function, the only critical number in the interval is 3.
We have: $f(3) = \frac{1}{6}$, $f(-1) = -\frac{1}{10}$, and $f(9) = \frac{1}{10}$.
So the absolute maximum value is 1/6 and the absolute minimum value is -0.1.