Math 75 **Fall 2003**

Practice test 2 - Answers

The actual exam will consist of 6 multiple choice questions and 6 regular problems. You will have 1 hour to complete the exam.

Multiple choice questions: circle the correct answer

1. Find the derivative of $\sqrt{2x}$.

   A. $\frac{2}{\sqrt{x}}$ **B.** $\frac{2}{\sqrt{2x}}$ **C.** $\frac{1}{2\sqrt{x}}$ **D.** $\frac{1}{\sqrt{2x}}$ **E.** $\frac{1}{2\sqrt{2x}}$

2. Find the fifth derivative of $\cos(x)$.

   A. $\sin(x)$ **B.** $-\sin(x)$ **C.** $\cos(x)$ **D.** $-\cos(x)$ **E.** 0

3. Evaluate $\lim_{x\to-\infty} e^x$.

   A. $-\infty$ **B.** 0 **C.** 1 **D.** $+\infty$ **E.** does not exist

4. Find the horizontal asymptote of $f(x) = \frac{x+2}{x-5}$.

   A. $x = -2$ **B.** $y = -2$ **C.** $y = 1$ **D.** $x = 5$ **E.** $y = 5$

5. Find the vertical asymptote of $f(x) = \frac{x+2}{x-5}$.

   A. $x = -2$ **B.** $y = -2$ **C.** $y = 1$ **D.** $x = 5$ **E.** $y = 5$

Regular problems: show all your work

6. Differentiate the following functions:

   (a) $f(x) = 3\cos(x^5) + \frac{\pi}{2}$

   $f'(x) = -3\sin(x^5)\cdot 5x^4 = -15x^4\sin(x^5)$

   (b) $f(x) = \cos(4)(x^3 - 3x)$

   $f'(x) = \cos(4)(3x^2 - 3)$

   (c) $g(x) = \frac{x^3 - 5}{\cos(-x)}$

   $$g'(x) = \frac{3x^2\cos x + (x^3 - 5)\sin x}{\cos^2 x}$$

   (d) $h(x) = \tan(x)\left(\frac{1}{\sqrt[4]{x^3}} + \frac{2}{x}\right)$

   $$h'(x) = \sec^2(x)\left(\frac{1}{\sqrt[4]{x^3}} + \frac{2}{x}\right) + \tan(x)\left(-\frac{3}{4}x^{-\frac{7}{4}} - \frac{2}{x^2}\right)$$

7. Find the first five derivatives of $g(x) = 27x^{4/3}$

   $g'(x) = 36x^{1/3}$

   $g''(x) = 12x^{-2/3}$

   $g'''(x) = -8x^{-5/3}$

   $g^{(4)}(x) = \dfrac{40}{3}x^{-8/3}$

   $g^{(5)}(x) = -\dfrac{320}{9}x^{-11/3}$

8. Find the points where the tangent line to the graph of $f(x) = x^5 - 80x$ is horizontal.

   The tangent line is horizontal when $f'(x) = 0$.

   $f'(x) = 5x^4 - 80 = 0$

   $5(x^4 - 16) = 0$

   $5(x^2 - 4)(x^2 + 4) = 0$

   $5(x-2)(x+2)(x^2+4) = 0$

   $x = 2$ and $x = -2$

   Thus the tangent line is horizontal at $(2, -128)$ and $(-2, 128)$.

9. Find an equation of the tangent line to $y = \sqrt{2x+3}$ at $(3, 3)$.

   The slope of the tangent line is equal to the derivative at 3.

   $y' = \dfrac{1}{2\sqrt{2x+3}} \cdot 2 = \dfrac{1}{\sqrt{2x+3}}$.

   $y'(3) = \dfrac{1}{3}$

   $y - 3 = \dfrac{1}{3}(x - 3)$

   $y = \dfrac{1}{3}x + 2$.

10. Find the linearization of $g(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.14}$.

    $g'(x) = \dfrac{1}{2\sqrt{x}}$

    $L(x) = g(1) + g'(1)(x-1) = 1 + \dfrac{1}{2}(x-1) = \dfrac{1}{2}x + \dfrac{1}{2}$

    $\sqrt{1.14} = g(1.14) \approx L(1.14) = \dfrac{1}{2} \cdot 1.14 + \dfrac{1}{2} = .57 + .5 = 1.07$

11. Consider the curve given by $x^3y^3 - 3xy^3 + 4y = 6$.

(a) Use implicit differentiation to find $y'(x)$.
$3x^2y^3 + x^3 3y^2 y' - 3y^3 - 3x3y^2y' + 4y' = 0$
$3x^3y^2y' - 9xy^2y' + 4y' = 3y^3 - 3x^2y^3$
$(3x^3y^2 - 9xy^2 + 4)y' = 3y^3 - 3x^2y^3$
$y' = \dfrac{3y^3 - 3x^2y^3}{3x^3y^2 - 9xy^2 + 4}$

(b) Check that the point $(2, 1)$ lies on this curve.

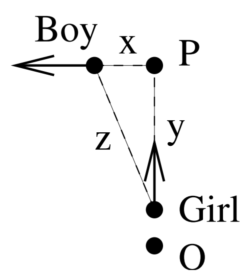
(c) $2^3 \cdot 1^3 - 3 \cdot 2 \cdot 1^3 + 4 \cdot 1 = 6$.

(d) What is the slope of the tangent line to this curve at $(2, 1)$?

(e) $y'(2) = \dfrac{3 \cdot 1^3 - 3 \cdot 2^2 \cdot 1^3}{3 \cdot 2^3 \cdot 1^2 - 9 \cdot 2 \cdot 1^2 + 4} = -0.9$.

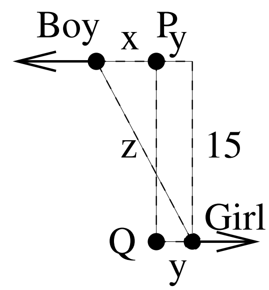
12. A boy starts walking west at 6 km/h from a point P. Five minutes later a girl starts walking (a) north (b) east at 4 km/h from a point 15 km due south from P. At what rate is the distance between the kids changing 45 km after the girl starts walking? Is the distance increasing or decreasing at this instant?

(a) Let x be the distance between the boy and the point P, let y be the distance between the girl and P, and let z be the distance between the boy and the girl.
Then $x^2 + y^2 = z^2$ where x, y, and z are functions of time.
Differentiating this equation with respect to t gives
$2xx' + 2yy' = 2zz'$
$xx' + yy' = zz'$



45 minutes after the girl started walking (and thus 50 minutes after the boy started walking), $x = 6 \cdot \frac{50}{60} = 5$, $y = 15 - 4 \cdot 4560 = 15 - 3 = 12$, and $z = \sqrt{5^2 + 12^2} = 13$.
x' is the rate of change of x, i.e. the speed of the boy, so $x' = 6$, and y' is the rate of change of y, i.e. negative the speed of the girl since y is decreasing, so $y' = -4$.
Therefore
$5 \cdot 6 + 12 \cdot (-4) = 13z$
Answer: $-\dfrac{18}{13}$, decreasing.

(b) Let x be the distance between the boy and the point P, let y be the distance between the girl and her starting point Q, and let z be the distance between the boy and the girl.



Then $(x+y)^2 + 15^2 = z^2$ (see the figure)
Differentiating this equation with respect to t gives
$2(x+y)(x'+y') = 2zz'$
$(x+y)(x'+y') = zz'$

45 minutes after the girl started walking (and thus 50 minutes after the boy started walking), $x = 6 \cdot \frac{50}{60} = 5$, $y = 4 \cdot 4560 = 3$, so $x + y = 8$, and $z = \sqrt{8^2 + 15^2} = 17$.
x' is the rate of change of x, i.e. the speed of the boy, so $x' = 6$, and y' is the rate of change of y, i.e. the speed of the girl, so $y' = 4$. Therefore
$(5+3)(6+4) = 17z'$
Answer: $\frac{80}{17}$, increasing.

13. A snowball is melting so that its radius is decreasing at a rate of 1 cm/min. Find the rate at which its volume is decreasing when the radius is 3 cm.

$V(t) = \frac{4}{3}\pi(r(t))^3$

$V'(t) = 4\pi(r(t))^2 r'(t)$

If $r' = -1$ and $r = 3$, $V'(t) = 4\pi 3^2 \cdot 1 = 36\pi$

Answer: 36π cm^3/min.

14. Find the critical numbers and local maxima and minima of
$f(x) = x^3 - 3x^2 + 5$.

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x)$ is positive if $x < 0$, negaive if $0 < x < 2$, and positive if $x > 2$.

Answer: Critical numbers: 0 and 2. Local maximum at 0, local minimum at 2.

15. Find the absolute maximum and minimum values of $f(x) = \sin x$ on the interval $\left[0, \frac{5\pi}{4}\right]$.

$f'(x) = \cos x$. Critical number: $\frac{\pi}{2}$. $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$.

Endpoints: 0 and $\frac{5\pi}{4}$. $f(0) = \sin(0) = 0$, $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Absolute maximum value is 1, absolute minimum value is $-\frac{1}{\sqrt{2}}$.

16. Evaluate the limits:

(a) $\lim\limits_{x\to\infty} \dfrac{2x^3+x-5}{5x^3-x^2} = \lim\limits_{x\to\infty} \dfrac{2+\frac{1}{x^2}-\frac{5}{x^3}}{5-\frac{1}{x}} = \dfrac{2}{5}$

(b) $\lim\limits_{x\to-\infty} \dfrac{x+1}{x^2+1} = \lim\limits_{x\to-\infty} \dfrac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}} = \dfrac{0}{1} = 0$

(c) $\lim\limits_{x\to\infty} \sqrt{x^2+3x-2}-x = \lim\limits_{x\to\infty} \dfrac{(\sqrt{x^2+3x-2}-x)(\sqrt{x^2+3x-2}+x)}{\sqrt{x^2+3x-2}+x}$

$= \lim\limits_{x\to\infty} \dfrac{x^2+3x-2-x^2}{\sqrt{x^2+3x-2}+x} = \lim\limits_{x\to\infty} \dfrac{3x-2}{\sqrt{x^2+3x-2}+x} = \lim\limits_{x\to\infty} \dfrac{3-\frac{2}{x}}{\frac{\sqrt{x^2+3x-2}}{x}+1}$

$= \lim\limits_{x\to\infty} \dfrac{3-\frac{2}{x}}{\frac{\sqrt{x^2+3x-2}}{\sqrt{x^2}}+1} = \lim\limits_{x\to\infty} \dfrac{3-\frac{2}{x}}{\sqrt{\frac{x^2+3x-2}{x^2}}+1} = \lim\limits_{x\to\infty} \dfrac{3-\frac{2}{x}}{\sqrt{1+\frac{3}{x}-\frac{2}{x^2}}+1} = \dfrac{3}{\sqrt{1}+1} = \dfrac{3}{2}$

(d) $\lim\limits_{x\to\infty} \tan x$ Does not exist

17. Let $f(x) = \dfrac{x}{(1+x)^2}$. Find the following:

(a) domain

$f(x)$ is defined for all x except -1, therefore, the domain is $(-\infty,-1)\cup(-1,+\infty)$.

(b) intercepts

To find x-intercepts, solve $f(x)=0$ for x: $\dfrac{x}{(1+x)^2} = 0$ gives $x=0$.

The y-intercept is $f(0) = \dfrac{0}{(1+0)^2} = 0$, so the only intercept is $(0,0)$.

(c) asymptotes

Horizontal asymptotes:

$\lim\limits_{x\to+\infty} \dfrac{x}{(1+x)^2} = \lim\limits_{x\to+\infty} \dfrac{x}{x^2+2x+1} = \lim\limits_{x\to+\infty} \dfrac{\frac{1}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = 0$

$\lim\limits_{x\to-\infty} \dfrac{x}{(1+x)^2} = \lim\limits_{x\to-\infty} \dfrac{x}{x^2+2x+1} = \lim\limits_{x\to-\infty} \dfrac{\frac{1}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = 0$

Thus there is one horizontal asymptote $y=0$.

Vertical asymptotes:

$\lim\limits_{x\to-1^+} \dfrac{x}{(1+x)^2} \left(= \dfrac{-1}{\text{small positive}}\right) = -\infty$

$\lim\limits_{x\to-1^-} \dfrac{x}{(1+x)^2} \left(= \dfrac{-1}{\text{small positive}}\right) = -\infty$

Thus $x=-1$ is a vertical asymptote.

(d) critical numbers

$f'(x) = \dfrac{1(1+x)^2 - x2(x+1)}{(1+x)^4} = \dfrac{(1+x)-2x}{(1+x)^3} = \dfrac{1-x}{(1+x)^3}.$

$f'(x)$ is not defined only at $x = -1$, but -1 is not in the domain of $f(x)$;
$f'(x) = 0$ at $x = 1$, so 1 is the only critical number.

(e) intervals of increase and decrease

$f(x)$ is increasing when $f'(x) > 0$, and decreasing when $f'(x) < 0$.

$$\begin{array}{lccc} & - & + & - \\ f'(x) & \multicolumn{3}{c}{\rule{6cm}{0.4pt}} \\ & -1 & & 1 \end{array}$$

Therefore $f(x)$ is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, +\infty)$.

(f) local and absolute maxima and minima

1 is a local maximum because the derivative changes from positive to negative at 1. Even though the derivative changes from neg. to pos. at -1, it is not a local minimum because $f(-1)$ is undefined.

There is no absolute minimum because $\lim\limits_{x \to -1^+} \dfrac{x}{(1+x)^2} = \lim\limits_{x \to -1^-} \dfrac{x}{(1+x)^2} = -\infty$.

1 is an absolute maximum because it is the only critical number, limits at infinity are 0, and there are no vertical asymptotes with $\lim\limits_{x \to a} \dfrac{x}{(1+x)^2} = \infty$. The absolute maximum value is $f(1) = \dfrac{1}{4}$.

(g) intervals of concavity

$$f''(x) = \frac{(-1)(1+x)^3 - (1-x)3(1+x)^2}{(1+x)^6} = \frac{-(1+x) - 3(1-x)}{(1+x)^4} = \frac{2x-4}{(1+x)^4}.$$

$f''(x) > 0$ when $x > 2$, and $f''(x) < 0$ when $x < 2$, therefore $f(x)$ is CU on $(2, +\infty)$, and CD on $(-\infty, -1) \cup (-1, 2)$.

(h) inflection points

$x = 2$ is the only inflection point ($f(x)$ changes from CD to CU at 2).

(i) sketch the graph of $f(x)$