Math 75  

Fall 2003

Practice test 2 - Answers

The actual exam will consist of 6 multiple choice questions and 6 regular problems. You will have 1 hour to complete the exam.

Multiple choice questions: circle the correct answer

1. Find the derivative of $\sqrt{2x}$.
   
   A. $\frac{2}{\sqrt{x}}$  
   B. $\frac{2}{\sqrt{2x}}$  
   C. $\frac{1}{2\sqrt{x}}$  
   D. $\frac{1}{\sqrt{2x}}$  
   E. $\frac{1}{2\sqrt{2x}}$

2. Find the fifth derivative of $\cos(x)$.
   
   A. $\sin(x)$  
   B. $-\sin(x)$  
   C. $\cos(x)$  
   D. $-\cos(x)$  
   E. 0

3. Evaluate $\lim_{{x \to -\infty}} e^x$.
   
   A. $-\infty$  
   B. $0$  
   C. $1$  
   D. $+\infty$  
   E. does not exist

4. Find the horizontal asymptote of $f(x) = \frac{x+2}{x-5}$.
   
   A. $x = -2$  
   B. $y = -2$  
   C. $y = 1$  
   D. $x = 5$  
   E. $y = 5$

5. Find the vertical asymptote of $f(x) = \frac{x+2}{x-5}$.
   
   A. $x = -2$  
   B. $y = -2$  
   C. $y = 1$  
   D. $x = 5$  
   E. $y = 5$

Regular problems: show all your work

6. Differentiate the following functions:

   (a) $f(x) = 3\cos(x^5) + \frac{\pi}{2}$

   $f'(x) = -15x^4\sin(x^5)$

   (b) $f(x) = \cos(4)(x^3 - 3x)$

   $f'(x) = \cos(4)(3x^2 - 3)$

   (c) $g(x) = \frac{x^3 - 5}{\cos(-x)}$

   $g'(x) = \frac{3x^2\cos x + (x^3 - 5)\sin x}{\cos^2 x}$

   (d) $h(x) = \tan(x) \left( \frac{1}{\sqrt{x^3}} + \frac{2}{x} \right)$

   $h'(x) = \sec^2(x) \left( \frac{1}{\sqrt{x^3}} + \frac{2}{x} \right) + \tan(x) \left( \frac{3}{4}x^{-\frac{7}{2}} - \frac{2}{x^2} \right)$
7. Find the first five derivatives of \( g(x) = 27x^{4/3} \)
   \[
g'(x) = 36x^{1/3}
\]
   \[
g''(x) = 12x^{-2/3}
\]
   \[
g'''(x) = -8x^{-5/3}
\]
   \[
g^{(4)}(x) = \frac{40}{3}x^{-8/3}
\]
   \[
g^{(5)}(x) = -\frac{320}{9}x^{-11/3}
\]

8. Find the points where the tangent line to the graph of \( f(x) = x^5 - 80x \) is horizontal.

   The tangent line is horizontal when \( f'(x) = 0 \).
   \[
f'(x) = 5x^4 - 80 = 0
\]
   \[
5(x^4 - 16) = 0
\]
   \[
5(x^2 - 4)(x^2 + 4) = 0
\]
   \[
5(x - 2)(x + 2)(x^2 + 4) = 0
\]
   \[x = 2 \text{ and } x = -2\]
   Thus the tangent line is horizontal at \((2, -128)\) and \((-2, 128)\).

9. Find an equation of the tangent line to \( y = \sqrt{2x + 3} \) at \((3, 3)\).

   The slope of the tangent line is equal to the derivative at 3.
   \[
y' = \frac{1}{2\sqrt{2x + 3}}, \quad 2 = \frac{1}{\sqrt{2x + 3}}
\]
   \[
y'(3) = \frac{1}{3}
\]
   \[y - 3 = \frac{1}{3}(x - 3)
\]
   \[y = \frac{1}{3}x + 2.
\]

10. Find the linearization of \( g(x) = \sqrt{x} \) at \( x = 1 \) and use it to approximate \( \sqrt{1.14} \).

   \[
g'(x) = \frac{1}{2\sqrt{x}}
\]
   \[
L(x) = g(1) + g'(1)(x - 1) = 1 + \frac{1}{2}(x - 1) = \frac{1}{2}x + \frac{1}{2}
\]
   \[
\sqrt{1.14} = g(1.14) \approx L(1.14) = \frac{1}{2} \cdot 1.14 + \frac{1}{2} = .57 + .5 = 1.07
\]
11. Consider the curve given by \( x^3y^3 - 3xy^3 + 4y = 6 \).

(a) Use implicit differentiation to find \( y'(x) \).
\[
 3x^2y^3 + x^3y^2y' - 3y^3 - 3x^3y^2y' + 4y' = 0
\]
\[
 3x^3y^2y' - 9xy^2y' + 4y' = 3y^3 - 3x^3y^3
\]
\[
 (3x^3y^2 - 9xy^2 + 4)y' = 3y^3 - 3x^2y^3
\]
\[
 y' = \frac{3y^3 - 3x^2y^3}{3x^3y^2 - 9xy^2 + 4}
\]

(b) Check that the point (2, 1) lies on this curve.

(c) \( 2^3 \cdot 1^3 - 3 \cdot 2 \cdot 1^3 + 4 \cdot 1 = 6 \).

(d) What is the slope of the tangent line to this curve at (2, 1)?
\[
 y'(2) = \frac{3 \cdot 1^3 - 3 \cdot 2^2 \cdot 1^3}{3 \cdot 2^3 \cdot 1^2 - 9 \cdot 2 \cdot 1^2 + 4} = -0.9.
\]

12. A boy starts walking west at 6 km/h from a point \( P \). Five minutes later a girl starts walking (a) north (b) east at 4 km/h from a point 15 km due south from \( P \). At what rate is the distance between the kids changing 45 km after the girl starts walking? Is the distance increasing or decreasing at this instant?

(a) Let \( x \) be the distance between the boy and the point \( P \), let \( y \) be the distance between the girl and \( P \), and let \( z \) be the distance between the boy and the girl.

Then \( x^2 + y^2 = z^2 \) where \( x \), \( y \), and \( z \) are functions of time.

Differentiating this equation with respect to \( t \) gives
\[
 2xx' + 2yy' = 2zz'
\]
\[
 xx' + yy' = zz'
\]

45 minutes after the girl started walking (and thus 50 minutes after the boy started walking), \( x = 6 \cdot \frac{50}{60} = 5 \), \( y = 15 - 4 \cdot 4560 = 15 - 3 = 12 \), and \( z = \sqrt{5^2 + 12^2} = 13 \).

\( x' \) is the rate of change of \( x \), i.e. the speed of the boy, so \( x' = 6 \), and \( y' \) is the rate of change of \( y \), i.e. negative the speed of the girl since \( y \) is decreasing, so \( y' = -4 \).

Therefore
\[
 5 \cdot 6 + 12 \cdot (-4) = 13z
\]

Answer: \( -\frac{18}{13} \), decreasing.
(b) Let $x$ be the distance between the boy and the point $P$, let $y$ be the distance between the girl and her starting point $Q$, and let $z$ be the distance between the boy and the girl.

Then $(x + y)^2 + 15^2 = z^2$ (see the figure)
Differentiating this equation with respect to $t$ gives
\[2(x + y)(x' + y') = 2zz'
\]
\[(x + y)(x' + y') = zz'
\]

45 minutes after the girl started walking (and thus 50 minutes after the boy started walking), $x = 6 \cdot \frac{50}{60} = 5$, $y = 4 \cdot 4560 = 3$, so $x + y = 8$, and $z = \sqrt{8^2 + 15^2} = 17$.
$x'$ is the rate of change of $x$, i.e. the speed of the boy, so $x' = 6$, and $y'$ is the rate of change of $y$, i.e. the speed of the girl, so $y' = 4$. Therefore
\[(5 + 3)(6 + 4) = 17z'
\]
Answer: $\frac{80}{17}$, increasing.

13. A snowball is melting so that its radius is decreasing at a rate of 1 cm/min. Find the rate at which its volume is decreasing when the radius is 3 cm.

\[V(t) = \frac{4}{3}\pi(r(t))^3\]
\[V'(t) = 4\pi(r(t))^2 r'(t)\]

If $r' = -1$ and $r = 3$, $V'(t) = 4\pi3^2 \cdot 1 = 36\pi$

Answer: $36\pi$ cm$^3$/min.

14. Find the critical numbers and local maxima and minima of
\[f(x) = x^3 - 3x^2 + 5.\]
\[f'(x) = 3x^2 - 6x = 3x(x - 2)\]
$f'(x)$ is positive if $x < 0$, negative if $0 < x < 2$, and positive if $x > 2$.

Answer: Critical numbers: 0 and 2. Local maximum at 0, local minimum at 2.

15. Find the absolute maximum and minimum values of $f(x) = \sin x$ on the interval $\left[0, \frac{5\pi}{4}\right]$.

\[f'(x) = \cos x.\] Critical number: $\frac{\pi}{2}$. $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$.

Endpoints: 0 and $\frac{5\pi}{4}$. $f(0) = \sin(0) = 0$, $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Absolute maximum value is 1, absolute minimum value is $-\frac{1}{\sqrt{2}}$. 

16. Evaluate the limits:

(a) \[ \lim_{x \to \infty} \frac{2x^3 + x - 5}{5x^3 - x^2} = \lim_{x \to \infty} \frac{2 + \frac{1}{x^2} - \frac{5}{x^3}}{5 - \frac{1}{x}} = \frac{2}{5} \]

(b) \[ \lim_{x \to -\infty} \frac{x + 1}{x^2 + 1} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x^2}}{1 + \frac{1}{x}} = 0 \]

(c) \[ \lim_{x \to \infty} \sqrt{x^2 + 3x - 2} - x = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 3x - 2} - x)(\sqrt{x^2 + 3x - 2} + x)}{\sqrt{x^2 + 3x - 2} + x} \]
\[ = \lim_{x \to \infty} \frac{x^2 + 3x - 2 - x^2}{\sqrt{x^2 + 3x - 2} + x} = \lim_{x \to \infty} \frac{3x - 2}{3x + x} = \lim_{x \to \infty} \frac{3 - \frac{2}{x}}{\sqrt{1 + \frac{3}{x} - \frac{2}{x^2} + 1}} = \frac{3}{2} \]

(d) \[ \lim_{x \to \infty} \tan x \quad \text{Does not exist} \]

17. Let \( f(x) = \frac{x}{(1 + x)^2} \). Find the following:

(a) domain
\[ f(x) \] is defined for all \( x \) except \(-1\), therefore, the domain is \((-\infty, -1) \cup (-1, +\infty)\).

(b) intercepts
To find \( x \)-intercepts, solve \( f(x) = 0 \) for \( x \):
\[ \frac{x}{(1 + x)^2} = 0 \] gives \( x = 0 \).

The \( y \)-intercept is \( f(0) = \frac{0}{(1 + 0)^2} = 0 \), so the only intercept is \((0, 0)\).

(c) asymptotes
Horizontal asymptotes:
\[ \lim_{x \to +\infty} \frac{x}{(1 + x)^2} = \lim_{x \to +\infty} \frac{x}{x^2 + 2x + 1} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{1 + \frac{2}{x} + \frac{1}{x^2}} = 0 \]

\[ \lim_{x \to -\infty} \frac{x}{(1 + x)^2} = \lim_{x \to -\infty} \frac{x}{x^2 + 2x + 1} = \lim_{x \to -\infty} \frac{\frac{1}{x}}{1 + \frac{2}{x} + \frac{1}{x^2}} = 0 \]

Thus there is one horizontal asymptote \( y = 0 \).

Vertical asymptotes:
\[ \lim_{x \to -1^+} \frac{x}{(1 + x)^2} = \frac{-1}{\text{small positive}} = -\infty \]
\[ \lim_{x \to -1^-} \frac{x}{(1 + x)^2} = \frac{-1}{\text{small positive}} = -\infty \]

Thus \( x = -1 \) is a vertical asymptote.

(d) critical numbers
\[ f'(x) = \frac{1(1 + x)^2 - x^2(x + 1)}{(1 + x)^4} = \frac{(1 + x) - 2x}{(1 + x)^3} = \frac{1 - x}{(1 + x)^3} \]
\[ f'(x) \] is not defined only at \( x = -1 \), but \(-1\) is not in the domain of \( f(x) \); 
\[ f'(x) = 0 \] at \( x = 1 \), so \( 1 \) is the only critical number.

(e) intervals of increase and decrease
\[ f(x) \] is increasing when \( f'(x) > 0 \), and decreasing when \( f'(x) < 0 \).
\[ f'(x) \]
\begin{array}{c|c|c}
-1 & + & - \\
\end{array}

Therefore \( f(x) \) is increasing on \((-1, 1)\) and decreasing on \((-\infty, -1)\) and \((1, +\infty)\).

(f) local and absolute maxima and minima
\( 1 \) is a local maximum because the derivative changes from positive to negative at \( 1 \). Even though the derivative changes from neg. to pos. at \(-1\), it is not a local minimum because \( f(-1) \) is undefined.

There is no absolute minimum because 
\[ \lim_{x \to -1^+} \frac{x}{(1+x)^2} = \lim_{x \to -1^-} \frac{x}{(1+x)^2} = -\infty. \]
\( 1 \) is an absolute maximum because it is the only critical number, limits at infinity are 0, and there are no vertical asymptotes with 
\[ \lim_{x \to a} \frac{x}{(1+x)^2} = \infty. \]
The absolute maximum value is \( f(1) = \frac{1}{4} \).

(g) intervals of concavity
\[ f''(x) = \frac{(-1)(1+x)^3 - (1-x)3(1+x)^2}{(1+x)^6} = \frac{-(1+x) - 3(1-x)}{(1+x)^4} = \frac{2x-4}{(1+x)^4}. \]
\[ f''(x) > 0 \] when \( x > 2 \), and \( f''(x) < 0 \) when \( x < 2 \), therefore \( f(x) \) is CU on \((2, +\infty)\), and CD on \((-\infty, -1) \cup (-1, 2)\).

(h) inflection points
\( x = 2 \) is the only inflection point \((f(x) \text{ changes from CD to CU at } 2)\).

(i) sketch the graph of \( f(x) \)