

Practice test 2 - Answers

The actual exam will consist of 6 multiple choice questions and 6 regular problems.
You will have 1 hour to complete the exam.

Multiple choice questions: circle the correct answer

- Find the derivative of $\sqrt{2x}$.
 A. $\frac{2}{\sqrt{x}}$ B. $\frac{2}{\sqrt{2x}}$ C. $\frac{1}{2\sqrt{x}}$ **D. $\frac{1}{\sqrt{2x}}$** E. $\frac{1}{2\sqrt{2x}}$
- Find the fifth derivative of $\cos(x)$.
 A. $\sin(x)$ **B. $-\sin(x)$** C. $\cos(x)$ D. $-\cos(x)$ E. 0
- Evaluate $\lim_{x \rightarrow -\infty} e^x$.
 A. $-\infty$ **B. 0** C. 1 D. $+\infty$ E. does not exist
- Find the horizontal asymptote of $f(x) = \frac{x+2}{x-5}$.
 A. $x = -2$ B. $y = -2$ **C. $y = 1$** D. $x = 5$ E. $y = 5$
- Find the vertical asymptote of $f(x) = \frac{x+2}{x-5}$.
 A. $x = -2$ B. $y = -2$ C. $y = 1$ **D. $x = 5$** E. $y = 5$

Regular problems: show all your work

6. Differentiate the following functions:

$$(a) f(x) = 3 \cos(x^5) + \frac{\pi}{2}$$

$$f'(x) = -3 \sin(x^5) \cdot 5x^4 = -15x^4 \sin(x^5)$$

$$(b) f(x) = \cos(4)(x^3 - 3x)$$

$$f'(x) = \cos(4)(3x^2 - 3)$$

$$(c) g(x) = \frac{x^3 - 5}{\cos(-x)}$$

$$g'(x) = \frac{3x^2 \cos x + (x^3 - 5) \sin x}{\cos^2 x}$$

$$(d) h(x) = \tan(x) \left(\frac{1}{\sqrt[4]{x^3}} + \frac{2}{x} \right)$$

$$h'(x) = \sec^2(x) \left(\frac{1}{\sqrt[4]{x^3}} + \frac{2}{x} \right) + \tan(x) \left(-\frac{3}{4}x^{-\frac{7}{4}} - \frac{2}{x^2} \right)$$

7. Find the first five derivatives of $g(x) = 27x^{4/3}$

$$g'(x) = 36x^{1/3}$$

$$g''(x) = 12x^{-2/3}$$

$$g'''(x) = -8x^{-5/3}$$

$$g^{(4)}(x) = \frac{40}{3}x^{-8/3}$$

$$g^{(5)}(x) = -\frac{320}{9}x^{-11/3}$$

8. Find the points where the tangent line to the graph of $f(x) = x^5 - 80x$ is horizontal.

The tangent line is horizontal when $f'(x) = 0$.

$$f'(x) = 5x^4 - 80 = 0$$

$$5(x^4 - 16) = 0$$

$$5(x^2 - 4)(x^2 + 4) = 0$$

$$5(x - 2)(x + 2)(x^2 + 4) = 0$$

$$x = 2 \text{ and } x = -2$$

Thus the tangent line is horizontal at $(2, -128)$ and $(-2, 128)$.

9. Find an equation of the tangent line to $y = \sqrt{2x + 3}$ at $(3, 3)$.

The slope of the tangent line is equal to the derivative at 3.

$$y' = \frac{1}{2\sqrt{2x+3}} \cdot 2 = \frac{1}{\sqrt{2x+3}}$$

$$y'(3) = \frac{1}{3}$$

$$y - 3 = \frac{1}{3}(x - 3)$$

$$y = \frac{1}{3}x + 2.$$

10. Find the linearization of $g(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.14}$.

$$g'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = g(1) + g'(1)(x - 1) = 1 + \frac{1}{2}(x - 1) = \frac{1}{2}x + \frac{1}{2}$$

$$\sqrt{1.14} = g(1.14) \approx L(1.14) = \frac{1}{2} \cdot 1.14 + \frac{1}{2} = .57 + .5 = 1.07$$

11. Consider the curve given by $x^3y^3 - 3xy^3 + 4y = 6$.

(a) Use implicit differentiation to find $y'(x)$.

$$3x^2y^3 + x^3 \cdot 3y^2y' - 3y^3 - 3x \cdot 3y^2y' + 4y' = 0$$

$$3x^3y^2y' - 9xy^2y' + 4y' = 3y^3 - 3x^2y^3$$

$$(3x^3y^2 - 9xy^2 + 4)y' = 3y^3 - 3x^2y^3$$

$$y' = \frac{3y^3 - 3x^2y^3}{3x^3y^2 - 9xy^2 + 4}$$

(b) Check that the point $(2, 1)$ lies on this curve.

(c) $2^3 \cdot 1^3 - 3 \cdot 2 \cdot 1^3 + 4 \cdot 1 = 6$.

(d) What is the slope of the tangent line to this curve at $(2, 1)$?

(e) $y'(2) = \frac{3 \cdot 1^3 - 3 \cdot 2^2 \cdot 1^3}{3 \cdot 2^3 \cdot 1^2 - 9 \cdot 2 \cdot 1^2 + 4} = -0.9$.

12. A boy starts walking west at 6 km/h from a point P . Five minutes later a girl starts walking (a) north (b) east at 4 km/h from a point 15 km due south from P . At what rate is the distance between the kids changing 45 km after the girl starts walking? Is the distance increasing or decreasing at this instant?

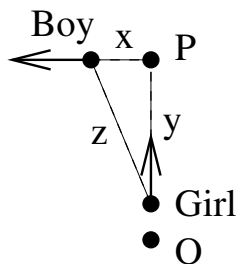
(a) Let x be the distance between the boy and the point P , let y be the distance between the girl and P , and let z be the distance between the boy and the girl.

Then $x^2 + y^2 = z^2$ where x , y , and z are functions of time.

Differentiating this equation with respect to t gives

$$2xx' + 2yy' = 2zz'$$

$$xx' + yy' = zz'$$



45 minutes after the girl started walking (and thus 50 minutes after the boy started walking), $x = 6 \cdot \frac{50}{60} = 5$, $y = 15 - 4 \cdot 45/60 = 15 - 3 = 12$, and $z = \sqrt{5^2 + 12^2} = 13$.

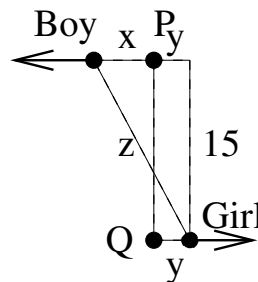
x' is the rate of change of x , i.e. the speed of the boy, so $x' = 6$, and y' is the rate of change of y , i.e. negative the speed of the girl since y is decreasing, so $y' = -4$.

Therefore

$$5 \cdot 6 + 12 \cdot (-4) = 13z'$$

Answer: $-\frac{18}{13}$, decreasing.

- (b) Let x be the distance between the boy and the point P , let y be the distance between the girl and her starting point Q , and let z be the distance between the boy and the girl.



Then $(x + y)^2 + 15^2 = z^2$ (see the figure)

Differentiating this equation with respect to t gives

$$2(x + y)(x' + y') = 2zz'$$

$$(x + y)(x' + y') = zz'$$

45 minutes after the girl started walking (and thus 50 minutes after the boy started walking), $x = 6 \cdot \frac{50}{60} = 5$, $y = 4 \cdot 4560 = 3$, so $x + y = 8$, and $z = \sqrt{8^2 + 15^2} = 17$.

x' is the rate of change of x , i.e. the speed of the boy, so $x' = 6$, and y' is the rate of change of y , i.e. the speed of the girl, so $y' = 4$. Therefore

$$(5 + 3)(6 + 4) = 17z'$$

Answer: $\frac{80}{17}$, increasing.

13. A snowball is melting so that its radius is decreasing at a rate of 1 cm/min. Find the rate at which its volume is decreasing when the radius is 3 cm.

$$V(t) = \frac{4}{3}\pi(r(t))^3$$

$$V'(t) = 4\pi(r(t))^2 r'(t)$$

$$\text{If } r' = -1 \text{ and } r = 3, V'(t) = 4\pi 3^2 \cdot 1 = 36\pi$$

Answer: 36π cm³/min.

14. Find the critical numbers and local maxima and minima of

$$f(x) = x^3 - 3x^2 + 5.$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$f'(x)$ is positive if $x < 0$, negative if $0 < x < 2$, and positive if $x > 2$.

Answer: Critical numbers: 0 and 2. Local maximum at 0, local minimum at 2.

15. Find the absolute maximum and minimum values of $f(x) = \sin x$ on the interval $\left[0, \frac{5\pi}{4}\right]$.

$$f'(x) = \cos x. \quad \text{Critical number: } \frac{\pi}{2}. \quad f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

$$\text{Endpoints: } 0 \text{ and } \frac{5\pi}{4}. \quad f(0) = \sin(0) = 0, \quad f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Absolute maximum value is 1, absolute minimum value is $-\frac{1}{\sqrt{2}}$.

16. Evaluate the limits:

$$(a) \lim_{x \rightarrow \infty} \frac{2x^3 + x - 5}{5x^3 - x^2} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x^2} - \frac{5}{x^3}}{5 - \frac{1}{x}} = \frac{2}{5}$$

$$(b) \lim_{x \rightarrow -\infty} \frac{x+1}{x^2+1} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x^2}} = \frac{0}{1} = 0$$

$$\begin{aligned} (c) \lim_{x \rightarrow \infty} \sqrt{x^2 + 3x - 2} - x &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 3x - 2} - x)(\sqrt{x^2 + 3x - 2} + x)}{\sqrt{x^2 + 3x - 2} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 + 3x - 2 - x^2}{\sqrt{x^2 + 3x - 2} + x} = \lim_{x \rightarrow \infty} \frac{3x - 2}{\sqrt{x^2 + 3x - 2} + x} = \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x}}{\frac{\sqrt{x^2 + 3x - 2}}{x} + 1} \\ &= \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x}}{\frac{\sqrt{x^2 + 3x - 2}}{\sqrt{x^2}} + 1} = \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x}}{\sqrt{1 + \frac{3}{x} - \frac{2}{x^2}} + 1} = \frac{3}{\sqrt{1} + 1} = \frac{3}{2} \end{aligned}$$

$$(d) \lim_{x \rightarrow \infty} \tan x \quad \text{Does not exist}$$

17. Let $f(x) = \frac{x}{(1+x)^2}$. Find the following:

(a) domain

$f(x)$ is defined for all x except -1 , therefore, the domain is $(-\infty, -1) \cup (-1, +\infty)$.

(b) intercepts

To find x -intercepts, solve $f(x) = 0$ for x : $\frac{x}{(1+x)^2} = 0$ gives $x = 0$.

The y -intercept is $f(0) = \frac{0}{(1+0)^2} = 0$, so the only intercept is $(0, 0)$.

(c) asymptotes

Horizontal asymptotes:

$$\lim_{x \rightarrow +\infty} \frac{x}{(1+x)^2} = \lim_{x \rightarrow +\infty} \frac{x}{x^2 + 2x + 1} = \lim_{x \rightarrow +\infty} \frac{\frac{1}{x}}{1 + \frac{2}{x} + \frac{1}{x^2}} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{x}{(1+x)^2} = \lim_{x \rightarrow -\infty} \frac{x}{x^2 + 2x + 1} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}}{1 + \frac{2}{x} + \frac{1}{x^2}} = 0$$

Thus there is one horizontal asymptote $y = 0$.

Vertical asymptotes:

$$\lim_{x \rightarrow -1^+} \frac{x}{(1+x)^2} \left(= \frac{-1}{\text{small positive}} \right) = -\infty$$

$$\lim_{x \rightarrow -1^-} \frac{x}{(1+x)^2} \left(= \frac{-1}{\text{small positive}} \right) = -\infty$$

Thus $x = -1$ is a vertical asymptote.

(d) critical numbers

$$f'(x) = \frac{1(1+x)^2 - x2(x+1)}{(1+x)^4} = \frac{(1+x) - 2x}{(1+x)^3} = \frac{1-x}{(1+x)^3}$$

$f'(x)$ is not defined only at $x = -1$, but -1 is not in the domain of $f(x)$;
 $f'(x) = 0$ at $x = 1$, so 1 is the only critical number.

(e) intervals of increase and decrease

$f(x)$ is increasing when $f'(x) > 0$, and decreasing when $f'(x) < 0$.

$$f'(x) \begin{array}{c} - \qquad \qquad + \qquad \qquad - \\ \hline -1 \qquad \qquad 1 \end{array}$$

Therefore $f(x)$ is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, +\infty)$.

(f) local and absolute maxima and minima

1 is a local maximum because the derivative changes from positive to negative at 1. Even though the derivative changes from neg. to pos. at -1 , it is not a local minimum because $f(-1)$ is undefined.

There is no absolute minimum because $\lim_{x \rightarrow -1^+} \frac{x}{(1+x)^2} = \lim_{x \rightarrow -1^-} \frac{x}{(1+x)^2} = -\infty$.

1 is an absolute maximum because it is the only critical number, limits at infinity are 0, and there are no vertical asymptotes with $\lim_{x \rightarrow a} \frac{x}{(1+x)^2} = \infty$. The absolute

maximum value is $f(1) = \frac{1}{4}$.

(g) intervals of concavity

$$f''(x) = \frac{(-1)(1+x)^3 - (1-x)3(1+x)^2}{(1+x)^6} = \frac{-(1+x) - 3(1-x)}{(1+x)^4} = \frac{2x-4}{(1+x)^4}$$

$f''(x) > 0$ when $x > 2$, and $f''(x) < 0$ when $x < 2$, therefore $f(x)$ is CU on $(2, +\infty)$, and CD on $(-\infty, -1) \cup (-1, 2)$.

(h) inflection points

$x = 2$ is the only inflection point ($f(x)$ changes from CD to CU at 2).

(i) sketch the graph of $f(x)$

