# MATH 75

# Test 3 - Solutions

**Multiple choice questions: circle the correct answer**

1. Which of the following in an antiderivative of $f(x) = 1 + \sin x$?

**A.** $\cos x$ **B.** $1 - \cos x$ **C.** $x + \cos x$ **(D.)** $1 + x - \cos x$ **E.** $-x\cos x$

2. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx =$

**A.** $-1$ **B.** $0$ **(C.)** $\frac{\pi}{4}$ **D.** $\frac{\pi}{2}$ **E.** $1$

3. $\displaystyle\int \sqrt{2x+1}\,dx =$

**A.** $\dfrac{1}{2\sqrt{2x+1}} + C$ **B.** $\dfrac{1}{\sqrt{2x+1}} + C$ **C.** $\dfrac{(2x+1)^{3/2}}{6} + C$

**(D.)** $\dfrac{(2x+1)^{3/2}}{3} + C$ **E.** $\dfrac{2(2x+1)^{3/2}}{3} + C$

4. If $f(x) = \displaystyle\int_0^x \sqrt{t^2+1}\,dt$, then $f'(x) =$

**A.** $\frac{\sqrt{x^2+1}}{2}$ **B.** $x\frac{\sqrt{x^2+1}}{2}$ **(C.)** $\sqrt{x^2+1}$ **D.** $x\sqrt{x^2+1}$ **E.** $\sqrt{x^2+1} - 1$

5. Use Newton's Method to approximate the root of $x^3 - 6x + 4 = 0$. Let $x_1 = 1$. Find $x_2$.

**A.** $-2$ **B.** $0$ **(C.)** $\frac{2}{3}$ **D.** $\frac{4}{3}$ **E.** $4$

6. Find the average value of the function $f(x) = \sin(2x)$ on the interval $\left[0, \frac{\pi}{4}\right]$.

**A.** $-\frac{2}{\pi}$ **B.** $-\frac{1}{2}$ **C.** $0$ **D.** $\frac{1}{2}$ **(E.)** $\frac{2}{\pi}$

**Regular problems: show all your work**

7. A box with a square base and open top must have a volume of 4,000 $\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

   *Let the length and the width of the box be $x$ and let the height be $y$. Then the surface area (which we have to minimize) is $A = x^2 + 4xy$, and the volume is $V = x^2y = 4000$. Solve the second equation for $y$: $y = \frac{4000}{x^2}$ and substitute into the first equation. Now $A(x) = x^2 + \frac{4x\cdot 4000}{x^2} = x^2 + \frac{16000}{x}$. To find the minimum, take the derivative and set it equal to 0: $A'(x) = 2x - \frac{16000}{x^2} = 0$.*
   $2x = \frac{16000}{x^2} \quad \Rightarrow \quad 2x^3 = 16000 \quad \Rightarrow \quad x^3 = 8000 \quad \Rightarrow \quad x = 20.$
   *The derivative changes from negative to positive at 20, so this is a minimum.*
   *Then $y = \frac{4000}{x^2} = \frac{4000}{400} = 10$.*
   *Thus the box must have length 20 cm, width 20 cm, and height 10cm.*

8. If $f'(x) = 10x^4 + 8x^3 + 6x^2 + 4$ and $f(-1) = 2$, find $f(x)$.

   *First find the general antiderivative: $f(x) = 2x^5 + 2x^4 + 2x^3 + 4x + c$. Now use the condition $f(-1) = 2$ to find $c$:*
   *$f(-1) = -2 + 2 - 2 - 4 + c = 2$, then $c = 8$. Therefore $f(x) = 2x^5 + 2x^4 + 2x^3 + 4x + 8$.*

9. Find the area of the region under the graph of $f(x) = \dfrac{1}{x^2}$ from $x = 1$ to $x = 2$.

   $$\textit{Area} = \int_1^2 \frac{1}{x^2}dx = \int_1^2 x^{-2}dx = \frac{x^{-1}}{-1}\Big|_1^2 = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

10. Find the volume of the solid obtained by rotating about the $x$-axis the region enclosed by $y = 1 - x^2$ and the $x$-axis.

    *The intersection points of the parabola and the $x$-axis are $x = 1$ and $x = -1$. Notice that the region is symmetric about the $y$-axis, so we will find the volume of the right half of the solid, and then multiply it by 2. Thus we will integrate with respect to $x$ from 0 to 1. Each cross-section is a disk, therefore* $\textit{Volume} = 2\left(\int_0^1 \pi(1-x^2)^2dx\right) = 2\pi\int_0^1 (1 - 2x^2 + x^4)dx =$
    $$2\pi\left(x - \frac{2x^2}{3} + \frac{x^5}{5}\right)\Big|_0^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi\frac{15 - 10 + 3}{15} = \frac{16\pi}{15}.$$

11. Find the area of the region enclosed by $x = 2y - y^2$ and $x = y^2 - 2y$.

    *First find the intersection points: $2y - y^2 = y^2 - 2y$*
    $4y - 2y^2 = 0$
    $2y(2 - y) = 0$
    *$y = 0$ and $y = 2$. Thus we will integrate with respect to $y$ from 0 to 2. If $y = 1$, then the first function has value 1, and the second function has value $-1$, thus the first function is the right one, and the second function is the left one. Now,*
    $$\textit{Area} = \int_0^2 (2y - y^2) - (y^2 - 2y)dy = \int_0^2 (4y - 2y^2)dy = \left(2y^2 - \frac{2y^3}{3}\right)\Big|_0^2 = 8 - \frac{16}{3} = \frac{8}{3}.$$

12. Find the volume of the solid obtained by rotating about $x = 1$ the region under the graph of $f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$.

    *Use cylindrical shells. The radius of a cylindrical shell is $x-1$, and the height is $\sqrt{x}$. Therefore*
    $$\textit{Volume} = 2\pi\int_1^4 (x-1)\sqrt{x}dx = 2\pi\int_1^4 (x^{3/2} - x^{1/2})dx = 2\pi\left(\frac{2x^{5/2}}{5} - \frac{2x^{3/2}}{3}\right)\Big|_1^4$$
    $$= 2\pi\left(\frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3}\right) = 2\pi\left(\frac{62}{5} - \frac{14}{3}\right) = 2\pi\frac{186 - 70}{15} = 2\pi\frac{116}{15} = \frac{232\pi}{15}.$$

- Good luck on the final!
- Have a nice winter break!