Math 75 **Fall 2003**

Curve Sketching

$$\text{Example. } f(x) = \frac{2x^2}{x^2-1}$$

Last time:

A. Domain. $f(x)$ is undefined when $x^2-1=0 \Leftrightarrow (x-1)(x+1)=0 \Leftrightarrow x=1,\ -1$. Thus Domain$(f) = (-\infty,-1) \cup (-1,1) \cup (1,+\infty)$.

B. Intercepts. y-intercept: $y(0)=0$.

x-intercepts: $f(x)=0 \Leftrightarrow \dfrac{2x^2}{x^2-1}=0 \Leftrightarrow x=0$.

Thus $(0,0)$ is the only intercept.

C. Symmetry. $\dfrac{2(-x)^2}{(-x)^2-1} = \dfrac{2x^2}{x^2-1} = 0$

i.e. $f(-x)=f(x)$, i.e. $f(x)$ is even, therefore the graph is symmetric about the y-axis.

D. Asymptotes.

horizontal:
$$\left.\begin{aligned} &\lim_{x\to+\infty}\frac{2x^2}{x^2-1} = \lim_{x\to+\infty}\frac{2}{1-\frac{1}{x^2}} = 2 \\ &\lim_{x\to-\infty}\frac{2x^2}{x^2-1} = \lim_{x\to-\infty}\frac{2}{1-\frac{1}{x^2}} = 2 \end{aligned}\right\} \Rightarrow\ y=2 \text{ is a horizontal asymptote.}$$

vertical:
$$\left.\begin{aligned} &\lim_{x\to1^+}\frac{2x^2}{x^2-1}\left(=\frac{2}{+0}\right) = +\infty \\ &\lim_{x\to1^-}\frac{2x^2}{x^2-1}\left(=\frac{2}{-0}\right) = -\infty \\ &\lim_{x\to-1^+}\frac{2x^2}{x^2-1}\left(=\frac{2}{-0}\right) = -\infty \\ &\lim_{x\to-1^-}\frac{2x^2}{x^2-1}\left(=\frac{2}{+0}\right) = +\infty \end{aligned}\right\} \Rightarrow\ x=1 \text{ and } x=-1 \text{ are vertical asymptotes.}$$

E. Increase / decrease.

$f'(x) = \dfrac{4x(x^2-1)-2x^2\cdot 2x}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$.

since $(x^2-1)^2>0$ for $x\neq\pm1$,

$f'(x)>0$ when $x<0 \Rightarrow f(x)$ is increasing on $(-\infty,-1)$ and on $(-1,0)$.

$f'(x)<0$ when $x>0 \Rightarrow f(x)$ is decreasing on $(0,1)$ and on $(1,+\infty)$.

F. Max / min.

$f'(x)=0 \Rightarrow \dfrac{2x^2}{x^2-1}=0 \Rightarrow x=0$.

At 0, $f'(x)$ changes from + to −, so 0 is a local maximum.

$f'(x)$ does not exist when $x^2-1=0 \Rightarrow x=\pm1$, but 1 and −1 are not in the domain of $f(x)$, so they are not critical numbers.

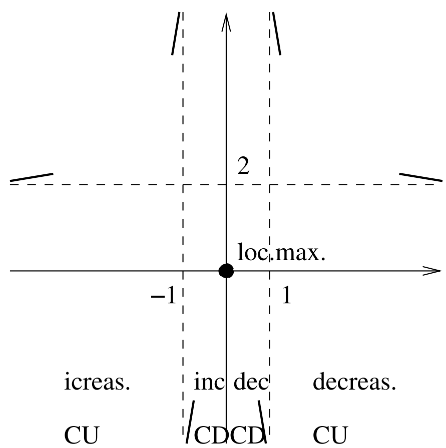
G. Concavity and inflection points.

$$f''(x) = \frac{-4(x^2-1)^2 - (-4x)2(x^2-1)2x}{(x^2-1)^4} = \frac{(x^2-1)[-4(x^2-1) + 4x \cdot 2 \cdot 2x]}{(x^2-1)^4} =$$
$$= \frac{-4x^2 + 4 + 16x^2}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3}.$$

since $12x^2 + 4 > 0$ for all x,
$f''(x) > 0$ when $x^2 - 1 > 0 \Rightarrow f(x)$ is CU on $(-\infty, -1)$ and on $(1, +\infty)$.
$f''(x) < 0$ when $x^2 - 1 < 0 \Rightarrow f(x)$ is CD on $(-1, 1)$.
$f(x)$ changes the direction of concavity at ± 1, but 1 and -1 are not in the domain of $f(x)$, so there are no inflection points.

H. Graph.

first sketch the asymptotes (also notice that when x is large or large negative, $f(x) = \frac{2x^2}{x^2-1} = \frac{2}{1-\frac{1}{x^2}}$ is slightly bigger than 2 because $1 - \frac{1}{x^2}$ is slightly smaller than 1, so the graph approaches the horizontal asymptote from above), plot the intercepts (only $(0,0)$ in our case), maxima ($(0,0)$ again), minima (none), and inflection points (none), and indicate the intevals of increase, decrease, and concavity:



now sketch the graph: