Curve Sketching

Example. \( f(x) = \frac{2x^2}{x^2 - 1} \)

Last time:
A. Domain. \( f(x) \) is undefined when \( x^2 - 1 = 0 \iff (x - 1)(x + 1) = 0 \iff x = 1, -1. \) Thus Domain \( f = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty). \)

B. Intercepts. \( y \)-intercept: \( y(0) = 0. \)
\( x \)-intercepts: \( f(x) = 0 \iff \frac{2x^2}{x^2 - 1} = 0 \iff x = 0. \) Thus \( (0,0) \) is the only intercept.

C. Symmetry. \( \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} = 0 \)
\( i.e. \ f(-x) = f(x), \ i.e. \ f(x) \) is even, therefore the graph is symmetric about the \( y \)-axis.

D. Asymptotes.

horizontal: \( \lim_{x \to \infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \infty} \frac{2}{1 - \frac{1}{x^2}} = 2 \)
\( \Rightarrow y = 2 \) is a horizontal asymptote.

vertical: \( \lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \frac{2}{+0} = +\infty \)
\( \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = \frac{2}{-0} = -\infty \)
\( \Rightarrow x = 1 \) and \( x = -1 \) are vertical asymptotes.

E. Increase / decrease.
\( f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}. \)
Since \( (x^2 - 1)^2 > 0 \) for \( x \neq \pm 1, \)
\( f'(x) > 0 \) when \( x < 0 \) \( \Rightarrow \ f(x) \) is increasing on \( (-\infty, -1) \) and on \( (-1, 0). \)
\( f'(x) < 0 \) when \( x > 0 \) \( \Rightarrow \ f(x) \) is decreasing on \( (0,1) \) and on \( (1, +\infty). \)

F. Max / min.
\( f'(x) = 0 \iff \frac{2x^2}{x^2 - 1} = 0 \iff x = 0. \)
At \( 0, \ f'(x) \) changes from + to −, so 0 is a local maximum.
\( f'(x) \) does not exist when \( x^2 - 1 = 0 \iff x = \pm 1, \) but 1 and \(-1\) are not in the domain of \( f(x) \), so they are not critical numbers.
G. Concavity and inflection points.

\[
f''(x) = \frac{-4(x^2 - 1)^2 - (-4x^2)(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{(x^2 - 1)[-4(x^2 - 1) + 4x \cdot 2 \cdot 2x]}{(x^2 - 1)^4} = \frac{-4x^2 + 4 + 16x^2}{(x^2 - 1)^3} = \frac{12x^2 + 4}{(x^2 - 1)^3}.
\]

Since \(12x^2 + 4 > 0\) for all \(x\),

\(f''(x) > 0\) when \(x^2 - 1 > 0\) \(\Rightarrow\) \(f(x)\) is CU on \((-\infty, -1)\) and on \((1, +\infty)\).

\(f''(x) < 0\) when \(x^2 - 1 < 0\) \(\Rightarrow\) \(f(x)\) is CD on \((-1, 1)\).

\(f(x)\) changes the direction of concavity at \(\pm 1\), but 1 and \(-1\) are not in the domain of \(f(x)\), so there are no inflection points.

H. Graph.

First sketch the asymptotes (also notice that when \(x\) is large or large negative, \(f(x) = \frac{2x^2}{x^2 - 1} = \frac{2}{1 - \frac{1}{x^2}}\) is slightly bigger than 2 because \(1 - \frac{1}{x^2}\) is slightly smaller than 1, so the graph approaches the horizontal asymptote from above), plot the intercepts (only \((0, 0)\) in our case), maxima ((0, 0) again), minima (none), and inflection points (none), and indicate the intervals of increase, decrease, and concavity:

\[
\begin{array}{ccc}
\text{inc} & \text{decrease} & \text{concavity}
\end{array}
\]

\[
\begin{array}{ccc}
\text{CU} & \text{CD} & \text{CU}
\end{array}
\]

Now sketch the graph: