Math 75 **Maria Voloshina**

Practice Final

The exam will consist of 25 multiple choice questions.

You will have 2 hours to complete the exam.

1. If $f(x) = (\sqrt[3]{x})^2$, then $\dfrac{df}{dx} =$

   (a) $\frac{2}{3}x^{5/3}$

   (b) $\frac{2}{3x^{1/3}}$ **correct**

   (c) $\frac{2}{3x^{-1/3}}$

   (d) $\frac{3}{5}x^{5/3}$

   (e) $\frac{3}{2}\sqrt{x}$

   Rewrite the function as $f(x) = x^{2/3}$, *then* $\dfrac{df}{dx} = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$

2. The vertical and horizontal asymptotes for the function $f(x) = \dfrac{3-x^2}{x^2-9}$ are

   (a) $x = 3$, $x = -3$, $y = -1$ **correct**

   (b) $x = 3$, $y = -1$

   (c) $x = -1$, $y = 3$, $y = -3$

   (d) $x = -1$, $y = -3$

   (e) $x = 3$, $x = -3$

   $f(x)$ *is undefined at* $x = 3$ *and* $x = -3$, *so check the limits at these points:*
   $\lim_{x\to 3^+} \dfrac{3-x^2}{x^2-9} = \lim_{x\to 3^+} \dfrac{3-x^2}{(x-3)(x+3)} \left(= \dfrac{appr.\ -6}{(small\ pos.)(appr.\ 9)}\right) = -\infty$, *so* $x = 3$ *is a vertical asymptote.*
   $\lim_{x\to -3^+} \dfrac{3-x^2}{x^2-9} = \lim_{x\to -3^+} \dfrac{3-x^2}{(x-3)(x+3)} \left(= \dfrac{appr.\ -6}{(-6)(small.\ pos.)}\right) = +\infty$, *so* $x = -3$ *is another vertical asymptote.*
   Limits at infinity:
   $\lim_{x\to +\infty} \dfrac{3-x^2}{x^2-9} = \lim_{x\to +\infty} \dfrac{\frac{3}{x^2}-1}{1-\frac{9}{x^2}} = -1$, *so* $y = 1$ *is a horizontal asymptote.*
   The limit as x *approaches* $-\infty$ *is similar.*

3. The derivative of $f(x) = \displaystyle\int_1^x \sin(1+t^4)dt$ is:

   (a) $\sin(1+x^4)$ **correct**

   (b) $4x^3\sin(1+x^4)$

   (c) $4x^3\cos(1+x^4)$

   (d) $-\sin(1+x^4)$

   (e) $-\cos(1+x^4)$

   Using part I of the fundamental theorem of calculus, $\left(\displaystyle\int_1^x \sin(1+t^4)dt\right)' = \sin(1+x^4)$.

4. The absolute minimum value of the function $f(x) = \sec x$ on the interval $[-1, 1]$ is

   (a) 0

   (b) -1

   (c) 1 **correct**

   (d) -2

   (e) 2

   $f'(x) = \sec x \tan x = \dfrac{\sin x}{\cos^2 x}$. *It is equal to 0 when the numerator is equal to 0, i.e. $sinx = 0$. This equation has only one solution in the interval $[-1, 1]$, namely, $x = 0$. Thus $x = 0$ is the only critical number.*
   $f(0) = \sec 0 = \dfrac{1}{\cos 0} = 1.$
   $f(1) = \sec 1 = \dfrac{1}{\cos 1} > 1$ *because* $\cos 1 < 1$ *(sketch the graph of* $\cos x$*!).*
   Similarly, $f(-1) > 1$, so 1 is the absolute minimum value of $f(x)$.

5. Which of the following is the linear approximation of the function $f(x) = \dfrac{1}{\sqrt{x}}$ near the number $a = 1$?

   (a) $2x - 1$

   (b) $-2x + 3$

   (c) $\frac{1}{2}x + \frac{1}{2}$

   (d) $-\frac{1}{2}x + 1$

   (e) $-\frac{1}{2}x + \frac{3}{2}$ **correct**

   The linear approximation is given by $L(x) = f(a) + f'(a)(x - a)$, and $a = 1$ in our case, so we need:
   $f(1) = 1,$
   $f'(x) = \left(x^{-1/2}\right)' = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}},$
   $f'(1) = -\frac{1}{2}.$
   Therefore $L(x) = 1 - \frac{1}{2}(x - 1) = -\frac{1}{2}x + \frac{3}{2}.$

6. Which of the following statements is true?

   (a) If f and g are increasing functions on an interval I, then fg is increasing on I.

   (b) If $f'(c) = 0$, then $f(x)$ has a local minimum or a local maximum at c.

   (c) If $f''(x) = 0$, then c is an inflection point of $f(x)$.

   (d) All continuous functions are differentiable.

   (e) All differentiable functions are continuous. **correct**

   Counterexamples for (a)-(d):

   (a) $f(x) = x$ and $g(x) = x$ are increasing everywhere, however, $f(x)g(x) = x^2$ is decreasing on $(-\infty, 0)$.

   (b) $f(x) = x^3$ has $f'(0) = 0$, but no minima or maxima.

   (c) $f(x) = x^4$ has $f''(0) = 0$, but no inflection points.

   (d) $f(x) = |x|$ is continuous but not differentiable at 0.

7. Find the value of k for which the function $f(x) = \begin{cases} \dfrac{x-9}{\sqrt{x}-3} & x \neq 9 \\ k & x = 9 \end{cases}$ is continuous at $x = 9$:

(a) 0
(b) -3
(c) 3
(d) 9
(e) 6 **correct**

$\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \lim_{x\to 9}(\sqrt{x}+3) = 6$, therefore to make the function continuous, we have to define $f(9) = 6$.

8. $\displaystyle\int_2^{10} |x-4|dx =$

(a) 0
(b) 16
(c) 18
(d) 20 **correct**
(e) 40

Since $|x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0, \text{ or } x \geq 0 \\ -x+4 & \text{if } x-4 < 0, \text{ or } x < 0 \end{cases}$,

$$\int_2^{10} |x-4|dx = \int_2^4 |x-4|dx + \int_4^{10} |x-4|dx = \int_2^4 (-x+4)dx + \int_4^{10} (x-4)dx = \left(-\frac{x^2}{2}+4x\right)\Bigg|_2^4 + \left(\frac{x^2}{2}-4x\right)\Bigg|_4^{10} = [(-8+16)-(-2+8)]+[(50-40)-(8-16)] = [8-6]+[10-(-8)] = 2+18 = 20.$$

Another way is to sketch the graph of $y = |x-4|$ and notice that the integral represents the total area of two triangles; the first one has base 2 and height 2, so its area is 2, and the second one has base 6 and height 6, so its area is 18. The total area is then 20.

9. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions of the poster will give the largest printed area?

(a) width $= 5\sqrt{5}$, height $= 6\sqrt{5}$
(b) width $= 6\sqrt{5}$, height $= 5\sqrt{5}$
(c) width $= 2\sqrt{30}$, height $= 3\sqrt{30}$ **correct**
(d) width $= 10$, height $= 15$
(e) width $= 15$, height $= 10$

Let the width of the poster be x, and let the height be y. Then the total area is $xy = 180$. The printed area is $A = (x-2)(y-3)$ (subtract the margins from x and y), and we want to maximize A. Solve the first equation for y: $y = \dfrac{180}{x}$, then A can be expressed as a function of x: $A = (x-2)(\frac{180}{x}-3) = 180 - 3x - \frac{360}{x} + 6 = 186 - 3x - \frac{360}{x}$. To find the maximum of this function, we take the derivative and set it equal to 0: $A'(x) = -3 + \frac{360}{x^2} = 0$, or

$\frac{360}{x^2} = 3$

$360 = 3x^2$

$x^2 = 120$

$x = 2\sqrt{30}$

Then $y = \frac{180}{2\sqrt{30}} = \frac{90}{\sqrt{30}} = 3\sqrt{30}$.

10. $\int (2 + 3\sin x + 4\cos x)dx =$

(a) $3\cos x - 4\sin x$

(b) $3\cos x - 4\sin x + C$

(c) $2x + 3\cos x - 4\sin x + C$

(d) $2x - 3\cos x + 4\sin x + C$ **correct**

(e) $2x + 3\cos x + 4\sin x + C$

Since an antiderivative of 2 is $2x$*, an antiderivative of* $\sin x$ *is* $-\cos x$*, and an antiderivative of* $\cos x$ *is* $\sin x$*,*
$\int (2 + 3\sin x + 4\cos x)dx = 2x + 3(-\cos x) + 4\sin x + C.$

11. The inflection points of the function $y = 2x^6 - 3x^5 - 10x^4 + 11$ are

(a) $(0, 11)$ only

(b) $(-1, 6)$ and $(2, 75)$ **correct**

(c) $(-1, 6)$, $(0, 1)$, and $(2, 75)$

(d) $(1, 0)$ only

(e) $(0, 11)$ and $(-1, 6)$

The second derivative must be 0 at an inflection point. We have $y' = 12x^5 - 15x^4 - 40x^3$*, and* $y'' = 60x^4 - 60x^3 - 120x^2$.
$60x^4 - 60x^3 - 120x^2 = 0$
$60x^2(x^2 - x - 2) = 0$
$60x^2(x - 2)(x + 1) = 0$
There are 3 roots: $x = -1$*,* $x = 0$*, and* $x = 2$*. Check that the second derivative is positive on* $(-\infty, -1)$*, negative on* $(-1, 0)$ *and* $(0, 2)$*, and positive again on* $(2, +\infty)$*, so it changes sign only at* -1 *and* 2*. Therefore only* $(-1, 6)$ *and* $(2, 75)$ *are inflection points.*

12. If $y = \cos(\cot x)$, then $\frac{dy}{dx} =$

(a) $-\sin(\cot x)$

(b) $-\sin(-\csc x \cot x)$

(c) $-\sin x \cot x - \cos x \csc x \cot x$

(d) $\frac{\cot x \cos x}{\sin x}$

(e) $\frac{\sin(\cot x)}{\sin^2 x}$ **correct**

Using the chain rule:
$\frac{dy}{dx} = (\cos(\cot x))' = -\sin(\cot x) \cdot (-\csc^2 x) = \sin(\cot x)(\csc^2 x) = \frac{\sin(\cot x)}{\sin^2 x}.$

13. Which of the following is equal to the area under the curve $y = |x^2 - 4|$ between $x = 0$ and $x = 4$?

(a) $\int_0^4 (x^2 - 4)dx$

(b) $-\int_0^4 (x^2 - 4)dx$

(c) $\int_2^4 (x^2 - 4)dx$

(d) $\int_0^2 (x^2 - 4)dx + \int_2^4 (4 - x^2)dx$

(e) $\int_0^2 (4 - x^2)dx + \int_2^4 (x^2 - 4)dx$ **correct**

Since $x^2 - 4$ is negative on $(0, 2)$, and positive on $(2, 4)$,

$|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x \geq 2 \\ -x^2 + 4 & \text{if } x < 2 \end{cases}$.

Then the area is $\int_0^4 |x^2 - 4|dx = \int_0^2 (-x^2 + 4)dx + \int_2^4 (x^2 - 4)dx$.

14. $\lim_{x \to 2} \frac{2x^2 - 5x + 2}{x^2 - x - 2} =$

(a) 0

(b) 1 **correct**

(c) 2

(d) ∞

(e) $-\infty$

$$\lim_{x \to 2} \frac{2x^2 - 5x + 2}{x^2 - x - 2} = \lim_{x \to 2} \frac{(2x - 1)(x - 2)}{(x - 2)(x + 1)} = \lim_{x \to 2} \frac{2x - 1}{x + 1} = \frac{3}{3} = 1.$$

15. $\int_0^3 x \sin(x^2 - 2)dx =$

(a) $\frac{1}{2}\cos(2) - \frac{1}{2}\cos(7)$ **correct**

(b) $-\frac{1}{2}\cos(7) - \frac{1}{2}\cos(2)$

(c) $1 - \cos(3)$

(d) $\cos(-2) - \cos(7)$

(e) $-\frac{1}{2}(\cos(3) - 1)$

Make the substitution $u = x^2 - 2$, then $\frac{du}{dx} = 2x$, so $\frac{1}{2}du = xdx$. Then, changing the limits of integration, we have

$$\int_0^3 x \sin(x^2 - 2)dx = \frac{1}{2}\int_{-2}^7 \sin(u)du = -\frac{1}{2}\cos(u)\Big|_{-2}^7 = -\frac{1}{2}cos(7) - \left(-\frac{1}{2}cos(-2)\right).$$

Another way is to evaluate the indefinite integral first (and change back to the original variable), and then use the old limits. Using the same substitution, we have

$\int x \sin(x^2 - 2)dx = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2 - 2) + C$. *Then*

$$\int_0^3 x \sin(x^2 - 2)dx = -\frac{1}{2}\cos(x^2 - 2)\Big|_0^3 = -\frac{1}{2}\cos(7) - \left(-\frac{1}{2}\cos(-2)\right).$$

16. If $F(x) = f(g(x))$, $f(1) = 0$, $f'(1) = 5$, $f'(2) = -4$, $g(1) = 2$, $g'(0) = -6$, and $g'(1) = 3$, then $F'(1) =$

   (a) -30

   (b) -24

   (c) -12 **correct**

   (d) 0

   (e) 15

   Using the chain rule, $F'(x) = (f(g(x)))' = f'(g(x))g'(x)$, *so* $F'(1) = f'(g(1))g'(1) = f'(2)\cdot 3 = (-4)3 = -12$.

17. What can be said about the roots of the equation $x^3 + x + 6 = 0$?

   (a) it has no real roots

   (b) it has exactly 1 real root between -3 and -1 **correct**

   (c) it has exactly 1 real root between -1 and 1

   (d) it has exactly 1 real root between 1 and 3

   (e) it has 3 real roots

   Let $f(x) = x^3 + x + 6$, *then* $f(-2) = -4 < 0$ *and* $f(0) = 6 > 0$, *so* $f(x)$ *has at least one real root. On the other hand,* $f'(x) = 3x^2 + 1 > 0$ *for all* x, *so* $f(x)$ *is increasing everywhere, so it cannot have more than one real root. Thus* $f(x)$ *has exactly one real root. Check the values of* $f(x)$ *at the endpoints of the given intervals:* $f(-3) = -24 < 0$, $f(-1) = 4 > 0$, *so* $f(x)$ *has a root between* -3 *and* -1.

18. $\displaystyle\int_{-1}^{1} x\sqrt{x^2+5}dx =$

   (a) $-\frac{4}{5}$

   (b) $-\frac{2}{5}$

   (c) 0 **correct**

   (d) $\frac{1}{15}$

   (e) $\frac{2}{5}$

   Make the substitution $u = x^2 + 5$, *then* $\frac{1}{2}du = xdx$, *and*

   $\displaystyle\int_{-1}^{1} x\sqrt{x^2+5}dx = \frac{1}{2}\int_{6}^{6}\sqrt{u}du = 0$. *Another way to see that the value of this integral is 0 is to notice that the integrand is an odd function, and the integral is from* $-a$ *to* a.

19. If $f(x) = \dfrac{1}{x^2}$ and $g(x) = \sqrt{x}$, then the domain of $f \circ g$ is

   (a) $(-\infty, \infty)$

   (b) $(0, \infty)$ **correct**

   (c) $[0, \infty)$

   (d) $(-\infty, 0) \cup (0, \infty)$

   (e) None of the above

   $f \circ g(x) = f(g(x)) = f(\sqrt{x}) = \frac{1}{(\sqrt{x})^2}$. *Since* $\sqrt{x}$ *is defined only for* $x \geq 0$, *and the denominator cannot be zero, the composite function is defined only for* $x > 0$.

20. If $f(x) = \pi^3 + \dfrac{x}{\sqrt{x}}$, then $f'(x) =$

(a) $3\pi^2 + \frac{1}{\frac{1}{2}x^{-1/2}}$

(b) $3\pi^2 + \frac{1}{2\sqrt{x}}$

(c) $\frac{1}{2\sqrt{x}}$ **correct**

(d) $\frac{\sqrt{x} - x\frac{1}{2}x^{-1/2}}{(\sqrt{x})^2}$

(e) $3\pi^2 + \frac{\sqrt{x} - x\frac{1}{2}x^{-1/2}}{(\sqrt{x})^2}$

π^3 is a constant, so its derivative is 0. Therefore
$f'(x) = \left(\pi^3 + \dfrac{x}{x^{1/2}}\right)' = \left(\pi^3 + x^{1/2}\right)' = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

21. The domain of the function $f(x) = \sqrt{\dfrac{1-x}{1+x}}$ is the set of all real numbers x for which:

(a) $x > 1$

(b) $x \geq 1$

(c) $-1 < x \leq 1$ **correct**

(d) $-1 < x$

(e) $x \neq -1$

First, the expression under the square root is defined only for $x \neq -1$ because the denominator cannot be zero. Second, the square root is only difened at nonnegative numbers, so $\dfrac{1-x}{1+x} \geq 0$. Since this fraction is 0 at $x = 1$ and undefined at $x = -1$, check each interval:
on $[-\infty, -1]$, $1 - x \geq 0$ and $1 + x \leq 0$, so $\dfrac{1-x}{1+x} \leq 0$.
on $[-1, 1]$, $1 - x \geq 0$ and $1 + x \geq 0$, so $\dfrac{1-x}{1+x} \geq 0$.
on $[1, +\infty]$, $1 - x \leq 0$ and $1 + x \geq 0$, so $\dfrac{1-x}{1+x} \leq 0$.
Since $x \neq -1$, the domain of $f(x)$ is $(-1, 1]$.

22. The graph of $y = x + \sin x$ has how many local maximums?

(a) 0 **correct**

(b) 1

(c) 2

(d) 3

(e) infinitely many

$y' = 1 - \cos x \geq 0$ for all x because $-1 \leq \cos x \leq 1$. So y is never decreasing. Therefore it has no local maximums (at a local maximum, a function changes from increasing to decreasing).

23. A particle moves along a straight line with equation of motion $s(t) = \sqrt{t+1}$. Find its average velocity over the time interval $[0, 3]$.

(a) $\frac{1}{3}$ **correct**

(b) $\frac{1}{\sqrt{3}}$

(c) $\frac{14}{9}$

(d) 1

(e) $-\frac{1}{12}$

The average velocity is the total distance traveled divided by the time elapsed. So we have $\dfrac{s(3)-s(0)}{3-0} = \dfrac{2-1}{3} = \dfrac{1}{3}$. *Another (longer) way is to find the velocity:* $v(t) = \frac{1}{2\sqrt{t+1}}$, *and then use the formula for the average value of a function (you will need to make the substitution* $u = x+1$ *to evaluate the integral).*

24. Evaluate $\displaystyle\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$.

(a) 0

(b) $\frac{1}{6}$ **correct**

(c) $\frac{1}{3}$

(d) 1

(e) ∞

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} = \lim_{x\to 7} \frac{(\sqrt{x+2}-3)(\sqrt{x+2}+3)}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)}$$
$$= \lim_{x\to 7} \frac{x-7}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{6}.$$

25. Let $\mathcal{R}$ be the region enclosed by the lines $y = \sqrt{x}$ and $y = \dfrac{x}{2}$. The volume of the solid formed by rotating $\mathcal{R}$ about the x-axis is

(a) $2\pi \displaystyle\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx$

(b) $\pi \displaystyle\int_0^4 \left(\left(\frac{x}{2} - x\right)^2\right) dx$

(c) $\pi \displaystyle\int_0^4 \left(x - \left(\frac{x}{2}\right)^2\right) dx$ **correct**

(d) $\pi \displaystyle\int_0^4 \left(\frac{x}{2} - \sqrt{x}\right)^2 dx$

(e) $2\pi \displaystyle\int_0^4 \left(\frac{x}{2} - \sqrt{x}\right)^2 dx$

The curves intersect when $\sqrt{x} = \frac{x}{2} \Rightarrow x = \frac{x^2}{4} \Rightarrow 4x = x^2 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0$ *2 roots:* $x = 0$ *and* $x = 4$. *Sketch the graphs! You'll see that on the interval* $[0, 2]$ *the curve* $y = \sqrt{x}$ *is above* $y = \dfrac{x}{2}$. *When we rotate the enclosed region about the x-axis, the cross-section through a point* $(x, 0)$ *and perpendicular to the x-axis is a ring with outer radius* $\sqrt{x}$ *and inner radius* $\dfrac{x}{2}$. *So the cross-sectional area is* $\pi(\sqrt{x})^2 - \pi\left(\frac{x}{2}\right)^2 = pi\left(x - \left(\frac{x}{2}\right)^2\right)$. *Therefore the volume is* $\pi \displaystyle\int_0^4 \left(x - \left(\frac{x}{2}\right)^2\right) dx$.