MATH 75
Test 2 - Solutions

Multiple choice questions: circle the correct answer

1. Find the derivative of the function $f(x) = \sin^3(x)$.

   A. $\cos^3(x)$ **B.** $3\sin^2(x)$ **C.** $3\cos^2(x)$ **D.** $3\sin^2(x)\cos(x)$ **E.** $-3\sin^2(x)\cos(x)$

2. Find the vertical and horizontal asymptotes for the function $f(x) = \dfrac{x^3 + 2x + 1}{x^3 - x}$.

   A. $x = 0,\ x = 1$ **B.** $x = 0,\ x = 1,\ y = 1$ **C.** $x = 1,\ y = 1$
   D. $x = -1,\ x = 0,\ x = 1,\ y = 1$ **E.** $x = -1,\ x = 0,\ x = 1,\ y = 0$

3. Evaluate the limit: $\displaystyle\lim_{x\to\infty} \frac{5x + 4}{x^2 + 2x - 1}$.

   A. $-\infty$ **B.** 0 **C.** 1 **D.** 5 **E.** ∞

4. If $f(x) = \sqrt{x}$, find $f''(4)$.

   A. $-\dfrac{1}{2}$ **B.** $-\dfrac{1}{32}$ **C.** 0 **D.** $\dfrac{1}{8}$ **E.** 2

5. The graph of $y = x^4 + 2x^2 + 5$ has how many inflection points?

   A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

6. Find the critical numbers of $y = 4x^3 - x^4$.

   A. 0 **B.** 2 **C.** 0 and 3 **D.** 0 and 4 **E.** no critical numbers

Regular problems: show all your work

7. Evaluate the limit: $\displaystyle\lim_{x\to-\infty} \frac{5x^2 + 4}{\sqrt{3x^2 + 2}}$

$$\lim_{x\to-\infty} \frac{5x^2+4}{\sqrt{3x^2+2}} = \lim_{x\to-\infty} \frac{\frac{5x^2+4}{x}}{\frac{\sqrt{3x^2+2}}{x}} = \lim_{x\to-\infty} \frac{5x+\frac{4}{x}}{\frac{\sqrt{3x^2+2}}{-\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{5x+\frac{4}{x}}{-\sqrt{\frac{3x^2+2}{x^2}}} = \lim_{x\to-\infty} \frac{5x+\frac{4}{x}}{-\sqrt{3+\frac{2}{x^2}}}$$

$$= \frac{-\infty}{-\sqrt{3}} = \infty.$$

8. Find the linear approximation of the function $f(x) = \cos(x)$ at $a = \dfrac{\pi}{2}$.

   $L(x) = f(a) + f'(a)(x - a)$

   $f(a) = \cos\left(\frac{\pi}{2}\right) = 0;\quad f'(x) = -\sin(x),\ so\ f'(a) = -\sin\left(\frac{\pi}{2}\right) = -1$

   Therefore $L(x) = 0 - \left(x - \frac{\pi}{2}\right) = \frac{\pi}{2} - x.$

9. Find the intervals of concavity of the function $f(x) = 4x^3 - 3x^2 - 1$.

   $f'(x) = 12x^2 - 6x$

   $f''(x) = 24x - 6.$

   $f''(x) > 0$ *if* $24x - 6 > 0$, *or* $24x > 6$, *or* $x > \frac{1}{4}$.

   $f''(x) < 0$ *if* $24x - 6 < 0$, *or* $24x < 6$, *or* $x < \frac{1}{4}$.

   Therefore $f(x)$ *is concave upward on* $\left(\frac{1}{4}, \infty\right)$ *and concave downward on* $\left(-\infty, \frac{1}{4}\right)$.

10. Find local maxima and minima of $f(x) = (x^2 - 1)^2$.

$f(x) = x^4 - 2x^2 + 1$

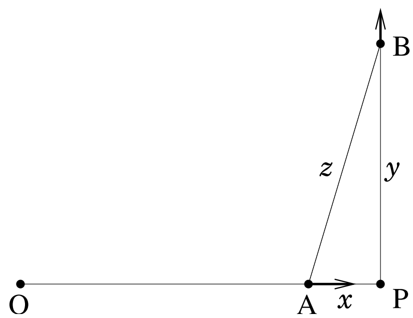
$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1)$.

$f'(x)$ is negative on $(-\infty, -1)$, positive on $(-1, 0)$, negative on $(0, 1)$, and positve again on $(1, \infty)$.

So $f'(x)$ changes from positive to negative at 0, and changes from negative to positive at -1 and at 1. Therefore $f(x)$ has a local maximum at 0, and local minuma at -1 and at 1.

11. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

Let x (or $x(t)$, since it is a function of t) be the distance between ship A and point P (see picture), then $x'(t) = -35$ since $x(t)$ is decreasing. Let y (or $y(t)$) be the distance between ship B and point P, then $y'(t) = 25$. Finally, let z (or $z(t)$) be the distance between the ships, then we want to find $z'(t)$.



$x^2 + y^2 = z^2$

$(x(t))^2 + (y(t))^2 = (z(t))^2$

Differentiate both sides with respect to t:

$2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t)$

$xx' + yy' = zz'$

At 4:00 PM, $x = 150 - 4 \cdot 35 = 10$, $y = 4 \cdot 25 = 100$, $z = \sqrt{10^2 + 100^2} = \sqrt{10100} = 10\sqrt{101}$.

$10(-35) + 100 \cdot 25 = 10\sqrt{101}z'$

$$z' = \frac{2150}{10\sqrt{101}} = \frac{215}{\sqrt{101}}$$

12. Find an equation of the tangent line to the curve $xy + 3x^2y^2 - 5x = 7$ at the point $(-1, 1)$.

Regard y as a function of x:

$xy(x) + 3x^2(y(x))^2 - 5x = 7$

Differentiate both sides with respect to x:

$y(x) + xy'(x) + 6x(y(x))^2 + 6x^2y(x)y'(x) - 5 = 0$

Rewrite to make it look simplier:

$y + xy' + 6xy^2 + 6x^2yy' - 5 = 0$

Plug in -1 for x and 1 for y:

$1 - y' - 6 + 6y' - 5 = 0$

Solve for y':

$5y' - 10 = 0 \ \Rightarrow \ 5y' = 10 \ \Rightarrow \ y' = 2$

An equation is then $y - 1 = 2(x + 1)$, or $y = 2x + 3$.