Review - 3

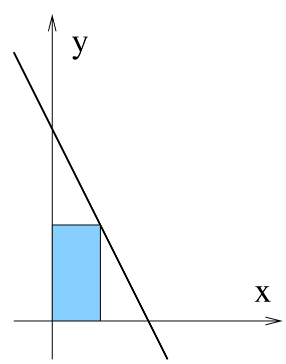
THEORY

Optimization problems

1. Draw a picture whenever possible.
2. Introduce notations. Assign symbols to the quantities that you need to find, and to the quantity that you want to maximize or minimize.
3. Express the quantity that is to be maximized or minimized in terms of some other quantities.
4. Use the given information to find relationships among the unknown quantities.
5. Express the quantity that is to be maximized or minimized in terms of just one variable.
6. To find a local maximum or minimum, differentiate the function from step 5, set the derivative equal to 0, and solve for the variable.
7. Find the values of all the required quantities.

Example. Find the point on the line $2x + y - 5 = 0$ and in the first quadrant such that the area of the rectangle bounded by the horizontal and the vertical lines through this point, the x-axis, and the y-axis, is as lagre as possible.

The area of the rectangle is $A = xy$, and since the point must lie on the above line, we have $y = 5 - 2x$. Then the area can be expressed as a function of one variable x, namely $A(x) = x(5 - 2x) = 5x - 2x^2$. Differentiate and set equal to 0: $A'(x) = 5 - 4x = 0$, so $x = 1.25$. Obviously, the derivative changes from positive to negative at 1.25, so we have a maximum. The y-coordinate of this point is $5 - 2 \cdot 1.25 = 2.5$.



There are 5 very good examples in the book (section 4.7). I strongly recommend you to read them all.

Newton's method

To approximate a root of an equation $f(x) = 0$, choose an appropriate initial approximation (e.g. sketch the graph of $f(x)$ and choose a number which is close to the root), call this initial approximation x_1. Then find x_2, x_3, etc. using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Antiderivatives and indefinite integrals

Def. $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$.

If $F(x)$ is an antiderivative of $f(x)$, then for any constant c, $F(x) + c$ is also an antiderivative of $f(x)$. Also, any antiderivative of $f(x)$ has the form $F(x) + c$. So the family of functions $F(x) + c$ is called the most general antiderivative of $f(x)$, and it is also called the indefinite integral of $f(x)$ denoted by $\int f(x)dx$.

Here is a table of indefinite integrals:

$f(x)$	$\int f(x)dx$
a	$ax + c$
$x^n, \quad n \neq -1$	$\frac{x^{n+1}}{n+1} + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\csc^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\csc x \cot x$	$-\csc x + c$

The following rules correspond to the sum, difference, and constant multiple rules for derivatives:

$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$

$\int (cf(x))dx = c \int f(x)dx$

If you are asked to give the particular anitiderivative that satisfies a certain initial contion, find the most general antiderivative (with c) first, and then use the initial condition to find the constant c. If you are given the second derivative $f''(x)$ (and possibly two initial condtions), then you'll have to antidifferentiate twice in order to find the function $f(x)$.

For example, to find $f(x)$ such that $f''(x) = \cos x$, $f'(0) = 1$ and $f(0) = 2$, write $f'(x) = \sin x + c$. The condition $f'(0) = 1$ implies that $c = 1$, then $f'(x) = \sin x + 1$, and $f(x) = -\cos x + x + d$. Now the condtion $f(0) = 2$ implies that $d = 3$, therefore $f(x) = -\cos x + x + 3$.

For an object moving along a straight line, its velocity function is an antiderivative of its acceleration function, and its position function is an antiderivative of its velocity function.

Note: acceleration due to gravity is approximately 9.8 m/s^2, or 32 ft/s^2.

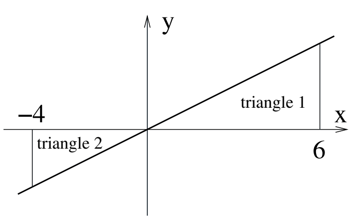
Area and the definite integral

Def. Let $f(x)$ be a continuous function on $[a, b]$. Divide $[a, b]$ into n subintervals of equal lenght $\Delta x = \dfrac{b-a}{n}$. Let $x_0 = a,\ x_1,\ x_2,\ \dots,\ x_{n-1},\ x_n = b$ be the endpoints of these subintervals. Choose a sample point in each subinterval: $x_i^* \in [x_{i-1}, x_i]$. Then the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum, and a limit of it as n approaches infinity is called the integral of $f(x)$ from a to b:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Note: If $f(x) > 0$, then the Riemann sum represents the sum of areas of approximating rectangles, and the integral $\int_a^b f(x)dx$ represents the area of the region under the graph of $f(x)$. If $f(x)$ takes on both positive and negative values, then $\int_a^b f(x)dx$ is the sum of the areas of regions under the graph of $f(x)$ and avove the x-axis minus the sum of the areas of regions above the graph of $f(x)$ and below the x-axis.

Example. The value of the integral $\int_{-4}^{6} \dfrac{x}{2}dx$ is the area of triangle 1 minus the area of triangle 2, i.e. $\dfrac{1}{2}\cdot 6\cdot 3 - \dfrac{1}{2}\cdot 4\cdot 2 = 9 - 4 = 5.$



Fundamental theorem of calculus

Part I. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

Part II. If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Example. $\displaystyle\int_{-4}^{6} \frac{x}{2}dx = \left.\frac{x^2}{4}\right|_{-4}^{6} = \frac{6^2}{4} - \frac{(-4)^2}{4} = \frac{36}{4} - \frac{16}{4} = 9 - 4 = 5.$

Cor. If $g(x) = \displaystyle\int_{a(x)}^{b(x)} f(t)dt$, then $g'(x) = f(b(x))b'(x) - f(a(x))a'(x)$

Example. $\displaystyle\frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{t}dt = \sqrt{\cos x}(-\sin x) - \sqrt{\sin x}\cos x = -\sqrt{\cos x}\sin x - \sqrt{\sin x}\cos x.$

The substitution rule

For an integral of the form $\int f(g(x))g'(x)dx$ make the substitution $u = g(x)$, then $du = g'(x)dx$, and

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Evaluate the integral $\int f(u)du$ (let $F(u)$ be an antiderivative of $f(u)$), and change back to the original variable x:

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + c = F(g(u)) + c$$

For a definite integral there are 2 ways to use substitution:

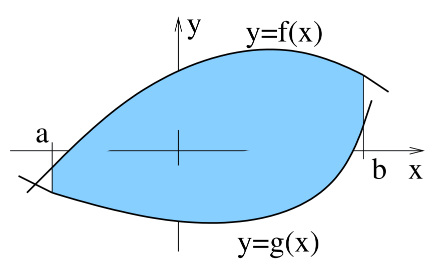
1. Evaluate the corresponding indefinite integral first, change back to the original variable, and then use the old limits of integration:
   If $\int f(g(x))g'(x)dx = \int f(u)du = F(u) + c = F(g(u)) + c$,
   then $\int_a^b f(g(x))g'(x)dx = F(g(u))\Big|_a^b = F(g(b)) - F(g(a))$.
2. Change the limits of integration as follows: since $u = g(x)$, $u = g(a)$ when $x = a$, and $u = g(b)$ when $x = b$, thus
   $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$.

Example. $\int_0^{\frac{\pi}{2}} \sin(2x)dx$

let $u = 2x$, then $du = 2dx$, or $\frac{du}{2} = dx$,
and the new limits of integration are $2 \cdot 0 = 0$ and $2 \cdot \frac{\pi}{2} = \pi$,
so the integral becomes
$\int_0^{\pi} \frac{\sin u}{2} du = \frac{1}{2}\int_0^{\pi} \sin u du = \frac{1}{2}(-\cos u)\Big|_0^{\pi} = \frac{1}{2}(-\cos\pi - (-\cos 0)) = \frac{1}{2}(1-(-1)) = 1.$

Areas between curves

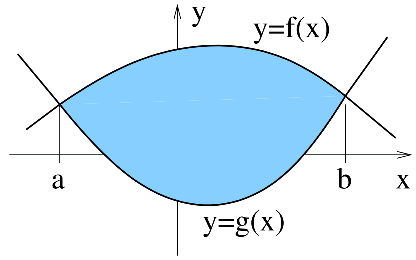
The area of the region bounded by two curves $y = f(x)$, $y = g(x)$ (where $f(x) \geq g(x)$), and vertical lines $x = a$ and $x = b$, is
$A = \int_a^b [f(x) - g(x)]dx$



To find the area of the region enclosed by 2 curves $y = f(x)$ and $y = g(x)$, you have to find the intersection points first. Set $f(x) = g(x)$ and solve for x. If you get 2 roots, the smaller one is the lower limit a, and the bigger one is the upper limit b.

Now, as before, integrate the difference (top function - bottom function) from a to b (let $f(x)$ be the top function, and let $g(x)$ be the bottom function):

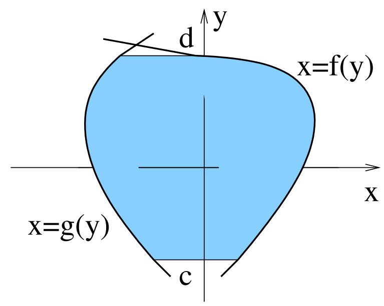
$$A = \int_a^b [f(x) - g(x)]dx$$



If you get 3 or more roots, it means that the region consists of 2 or more parts. Find the area of each part by integrating (top function - bottom function).

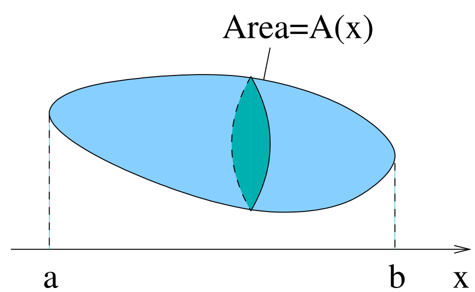
Sometimes it's better to regard x as a function of y (e.g. when the curves are given by equations $x = f(y)$ and $x = g(y)$). In this case, integrate (right function - left function) with respect to y:

$$A = \int_c^d [f(y) - g(y)]dy$$



Volumes using cross-sections

In general, the volume of a solid located between $x = a$ and $x = b$, is equal to the integral of the cross-sectional area $A(x)$ from a to b ($A(x)$ is the area of the cross-section through the point $(x, 0, 0)$ and perpendicular to the x-axis: $V = \int_a^b A(x)dx$



As with areas, it is sometimes better to regard x as a function of y and integrate with respect to y. If a solid is located between $y = c$ and $y = d$, then the volume is $V = \int_c^d A(y)dy$ where $A(y)$ is the cross-sectional area.

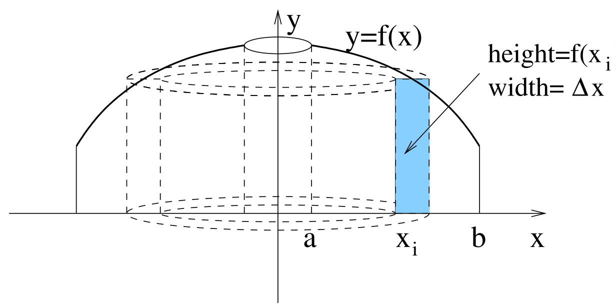
If the solid is obtained by rotating the region under the graph of a function $y = f(x)$ from $x = a$ to $x = b$ about the x-axis (such solids are called solids of revolution), then each cross-section is a disk with radius $r = f(x)$, so the cross-sectional area is $A(x) = \pi(f(x))^2$, and $V = \int_a^b \pi(f(x))^2 dx$.

If the solid is obtained by rotating the region between two curves, say $y = f(x)$ and $y = g(x)$, then every cross-section is a "washer" with outer radius $f(x)$ and inner radius $g(x)$, whose area is $A(x) = \pi(f(x))^2 - \pi(g(x))^2$.

Also, we can rotate the above regions about any horizontal or vertical line, the radii are then different (see next page). Don't memorize all the formulas on the next page, rather go though each of them and make sure you **understand** how to compute the volume in each case.

Volumes by cylindrical shells

$V = \int_a^b 2\pi x f(x) dx$



Here is an explanation of this formula: if we divide $[a, b]$ into n subintervals, and approximate the region under the graph of $f(x)$ by n rectangles, then the solid obtained by rotating the i^{th} rectangle about the y-axis is called a cylindrical shell.

Its base is a washer with circumference $2\pi x_i$ and thickness Δx, thus area $2\pi x_i \Delta x$, and its height is $f(x_i)$. Then the volume of the whole solid is approximately $\sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x$. Taking the limit as $n \to \infty$ gives the integral above.

Use this idea to find volumes of other solids, say, the solid obtained by rotating the region between $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ about the line $x = c$. Then the radius of a cylindrical shell is $x - c$, and the height is $f(x) - g(x)$, so the volume in this case is $V = \int_a^b 2\pi(x - c)(f(x) - g(x))dx$

As before, if the region is bounded by the curve $x = f(y)$ and the y-axis (or by two curves $x = f(y)$ and $x = g(y)$), and it is rotated about the x-axis, then you have to integrage with respect to y: $V = \int_a^b 2\pi y f(y) dy$.

Average value of a function

The average value of a function $f(x)$ on an interval $[a, b]$ is $f_{\text{ave}[a,b]} = \frac{1}{b-a}\int_a^b f(x)dx$. The meaning of the average value is that the area of the region under the graph of $f(x)$ is equal to the area of the rectangle with length $(b - a)$ and height $f_{\text{ave}[a,b]}$.