Math 75 **Maria Nogin**

Review - 1

THEORY

Useful formulas

$$(a+b)^2 = a^2 + 2ab + b^2, \quad (a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

Intervals

- $x \in (a,b) \iff a < x < b$
- $x \in [a,b) \iff a \le x < b$
- $x \in (a,b] \iff a < x \le b$
- $x \in [a,b] \iff a \le x \le b$
- $x \in (a,+\infty) \iff a < x$
- $x \in [a,+\infty) \iff a \le x$
- $x \in (-\infty,b) \iff x < b$
- $x \in (-\infty,b] \iff x \le b$

Absolute value

$$|x| = \begin{cases} x & \text{if} \quad x \ge 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

If a is a positive number,

- $|x| = a \iff x = -a$ or $x = a$
- $|x| < a \iff -a < x < a \iff x \in (-a,a)$
- $|x| \le a \iff -a \le x \le a \iff x \in [-a,a]$
- $|x| > a \iff x < -a$ or $x > a \iff x \in (-\infty,-a) \cup (a,+\infty)$
- $|x| \ge a \iff x \le -a$ or $x \ge a \iff x \in (-\infty,-a] \cup [a,+\infty)$

Coordinate geometry and lines

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The slope of the line that passes through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a horizontal line is equal to 0.
The slope of a vertical line is undefined.
An equation of the line that passes through the point $P(x_1, y_1)$ and has slope m is

$$y - y_1 = m(x - x_1) \quad \text{(point-slope equation)}$$

To find an equation of the line that passes through points $P(x_1, y_1)$ and $Q(x_2, y_2)$, first find its slope and then use the point-slope equation.

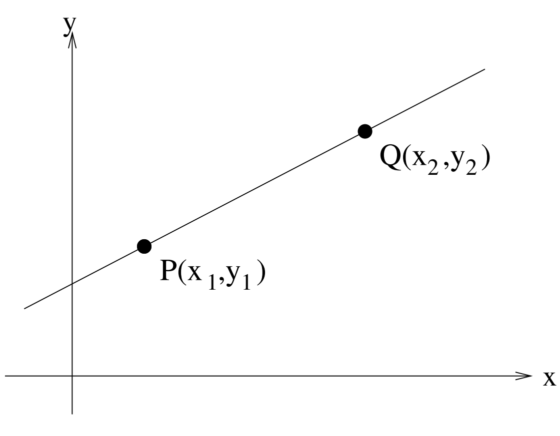
An equation of the line that has slope m and intersects the y-axis at the point $(0, b)$ is

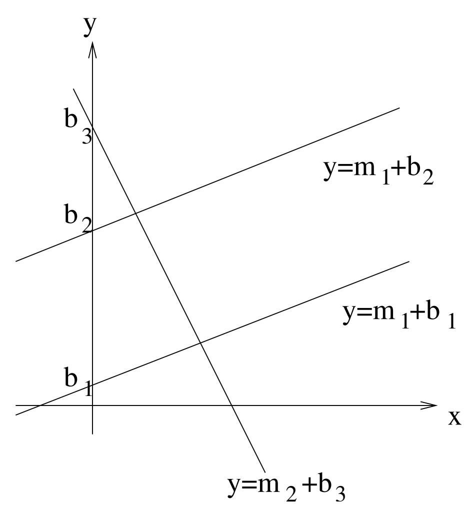
$$y = mx + b \quad \text{(slope-intercept equation)}$$

The equation of any line can be written in the general form:

$$ax + by + c = 0$$

Two non-vertical lines are parallel if and only if they have the same slope.
Two non-vertical lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 \cdot m_2 = -1$.





Trigonometry

$$180^{\text{O}} = \pi \text{ rad} \qquad\qquad 1^{\text{O}} = \frac{\pi}{180} \text{ rad}$$

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x, \qquad \tan(-x) = -\tan x$$

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x, \quad \tan(x + \pi) = \tan x$$

$$\sin(2x) = 2 \sin x \cos x, \qquad \cos(2x) = (\cos x)^2 - (\sin x)^2$$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0

$$\arcsin x = \sin^{-1} x = y \text{ s.t. } \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\arccos x = \cos^{-1} x = y \text{ s.t. } \cos y = x \text{ and } 0 \leq y \leq \pi$$

$$\arctan x = \tan^{-1} x = y \text{ s.t. } \tan y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Exponential functions

- Functions of the form a^x.
Laws of Exponents:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = (a^y)^x = a^{xy}$
- $\sqrt[x]{a} = a^{1/x}$
- $a^0 = 1$
- $a^1 = a$

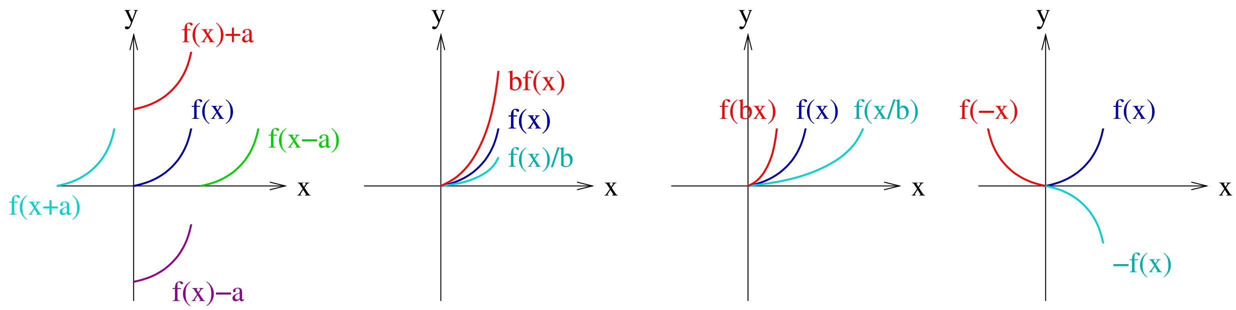
Domain and Range

Def. The domain of $f(x)$ is the set of all values of x for which $f(x)$ is defined.
Def. The range of $f(x)$ is the set of all values of $y = f(x)$.

Transformations of functions

Let $a > 0$ and $b > 1$. To obtain the graph of

- $y = f(x) + a$, shift the graph of $y = f(x)$ a units upward.
- $y = f(x) - a$, shift the graph of $y = f(x)$ a units downward.
- $y = f(x + a)$, shift the graph of $y = f(x)$ a units to the left.
- $y = f(x - a)$, shift the graph of $y = f(x)$ a units to the right.
- $y = bf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of b.
- $y = \frac{f(x)}{b}$, compress the graph of $y = f(x)$ vertically by a factor of b.
- $y = f(bx)$, compress the graph of $y = f(x)$ horizontally by a factor of b.
- $y = f(\frac{x}{b})$, stretch the graph of $y = f(x)$ horizontally by a factor of b.
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.



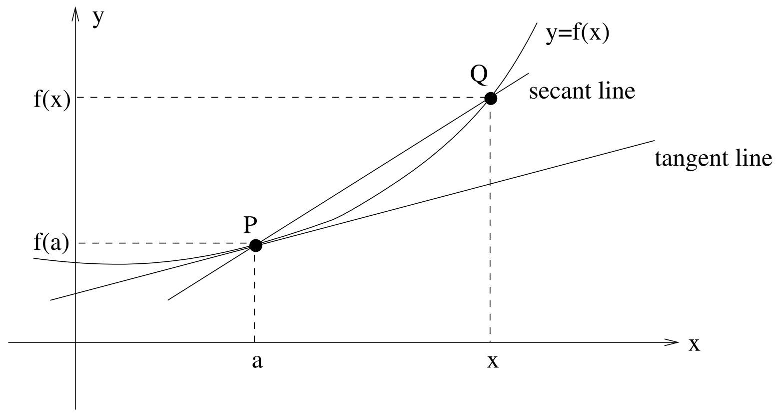
Composition of functions

$$(f \circ g)(x) = f(g(x))$$

Tangent line

A tangent line to a curve is a line that "touches" the curve. To find an equation of the tangent line to the curve $y = f(x)$ at the point P$(a, f(a))$, we have to know its slope. Let Q$(x, f(x))$ be another point on the curve. The slope of the secant line PQ is $m_{PQ} = \dfrac{f(x) - f(a)}{x - a}$. The slope of the tangent line at P is the limiting value of the slopes of secant lines PQ as Q approaches P, i.e.

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

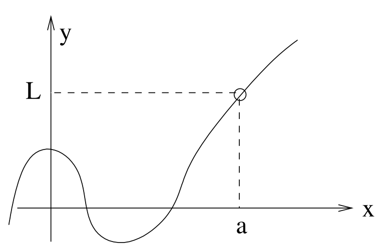


The limit of a function

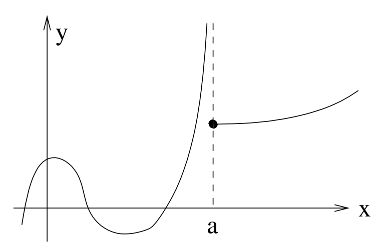
Def. We write $\lim_{x \to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a, but not equal to a.

Note: The function $f(x)$ may or may not be defined at the point a.

If the values of $f(x)$ do not approach any number as x approaches a, we say that the limit $\lim_{x \to a} f(x)$ does not exist.



$\lim_{x \to a} f(x) = L$



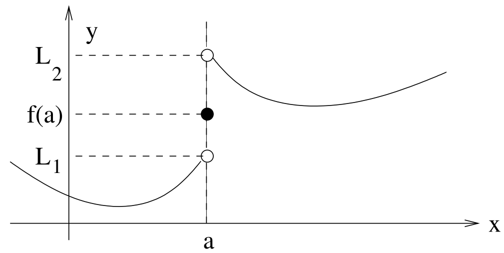
$\lim_{x \to a} f(x)$ DNE

Def. We write $\lim_{x \to a^-} f(x) = L$ and say "the limit of $f(x)$ as x approaches a from the left is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Def. We write $\lim_{x \to a^+} f(x) = L$ and say "the limit of $f(x)$ as x approaches a from the right is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a.

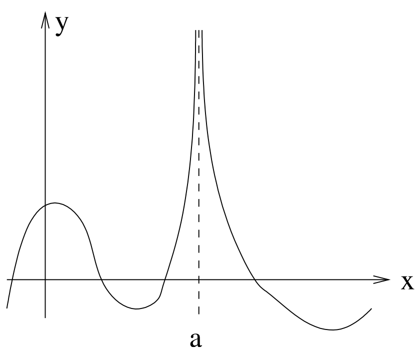
$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

$\lim\limits_{x\to a^-} f(x) = L_1$
$\lim\limits_{x\to a^+} f(x) = L_2$
$\lim\limits_{x\to a} f(x)$ DNE b/c $L_1 \neq L_2$

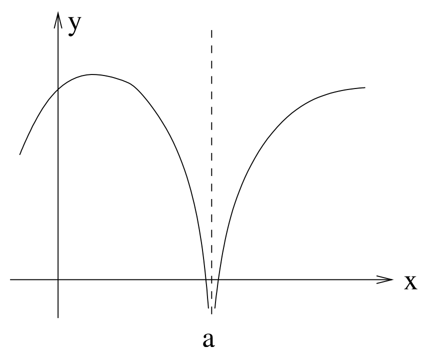


Def. $\lim\limits_{x\to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

Def. $\lim\limits_{x\to a} f(x) = -\infty$ means that the values of $f(x)$ can be make arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

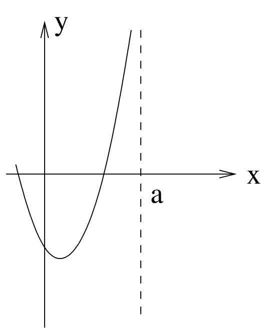


$\lim\limits_{x\to a} f(x) = \infty$

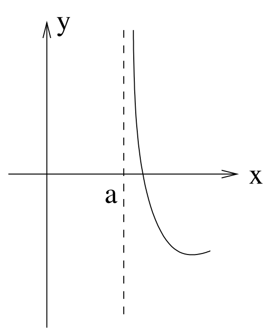


$\lim\limits_{x\to a} f(x) = -\infty$

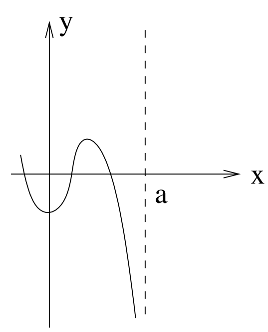
The definitions for one-sided infinite limits are similar. Here are the pictures:



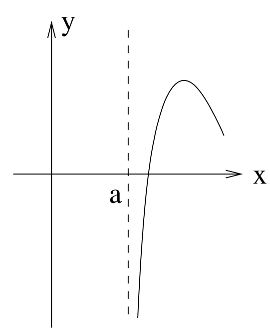
$\lim\limits_{x\to a^-} f(x) = \infty$



$\lim\limits_{x\to a^+} f(x) = \infty$



$\lim\limits_{x\to a^-} f(x) = -\infty$



$\lim\limits_{x\to a^+} f(x) = -\infty$

If at least one of these 4 conditions holds, the line $x = a$ is called a vertical asymptote of $y = f(x)$.

Limit laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^c = [\lim_{x\to a} f(x)]^c$
7. $\lim_{x\to a} \sqrt[c]{f(x)} = \lim_{x\to a} \sqrt[c]{f(x)}$
8. $\lim_{x\to a} c = c$
9. $\lim_{x\to a} x = a$
10. $\lim_{x\to a} x^c = a^c$
11. $\lim_{x\to a} \sqrt[c]{x} = \sqrt[c]{a}$

If $P(x)$ is a polynomial function (i.e. $P(x) = a_n x^n + a_{n-1}x^{n-1} \dots a_1 x + a_0$), then for any c, $\lim_{x\to c} P(x) = P(c)$.

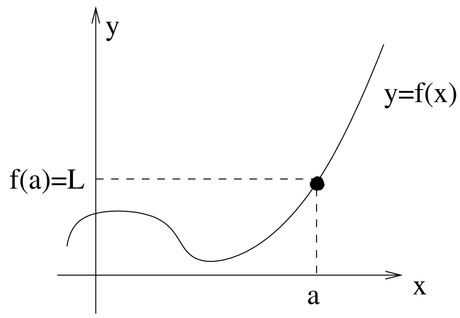
If $P(x)$ is a rational function (i.e. $P(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} \dots a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} \dots b_1 x + b_0}$), then for any c such that $P(x)$ is defined at c (i.e. the denominator is not equal to 0 at c), $\lim_{x\to c} P(x) = P(c)$.

Squeeze theorem. If $f(x) \leq g(x) \leq h(x)$ when x is near a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Continuity

Def. A function $f(x)$ is continuous at a number a if

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$



If $f(x)$ is not continuous at a, we say f is discontinuous at a.

Def. A function $f(x)$ is continuous on an interval if it is continuous at every number in the interval.

Theorem. All power, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions, as well as all their combinations and compositions are continuous everywhere in their domain (i.e. wherever they are defined).

Derivatives

Definition. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

(if this limit exists. In this case we say that $f(x)$ is differentiable at a.
If the limit does not exist, then $f(x)$ is not differentiable at a.)

The derivative of $f(x)$ is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The tangent line to $y = f(x)$ at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$.

Differentiation rules

1. Sum and difference rules $(f+g)' = f' + g'$, $(f-g)' = f' - g'$
2. Constant multiple rule $(cf)' = cf'$ for any constant c
3. Product rule $(fg)' = f'g + fg'$
4. Quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Derivatives of power functions

$$(x^n)' = nx^{n-1}, \qquad\qquad (c)' = 0$$