MATH 75
Test 2 - Solutions
June 16, 2005

Multiple choice questions: circle the correct answer

1. Find the derivative of \( f(x) = \sin(4x^2) \).
   \begin{align*}
   &A. \cos(4x^2) \quad B. \cos(8x^2) \quad C. 8x \cos(4x^2) \quad D. -4x^2 \cos(8x) \quad E. -\cos(x)(4x^2)
   \end{align*}

2. Find the vertical asymptotes of \( f(x) = \frac{1 - x^2}{x^2 - 4x} \).
   \begin{align*}
   &A. x = 0 \quad B. x = 4 \quad C. x = 0 \text{ and } x = 4 \quad D. y = -1 \quad E. y = -4
   \end{align*}

3. Evaluate the limit: \( \lim_{x \to \infty} \frac{x^2 + 10}{x^3 - 3} \).
   \begin{align*}
   &A. 0 \quad B. 1 \quad C. -\frac{10}{3} \quad D. \infty \quad E. -\infty
   \end{align*}

4. If \( f(t) = \frac{1}{x^2} \), find \( f''(-1) \).
   \begin{align*}
   &A. -6 \quad B. -2 \quad C. 0 \quad D. 2 \quad E. 6
   \end{align*}

5. How many inflection points does the function \( y = x + \frac{1}{x} \) have?
   \begin{align*}
   &A. 0 \quad B. 1 \quad C. 2 \quad D. 3 \quad E. \text{infinitely many}
   \end{align*}

6. Find the local minimum of \( y = x + \frac{1}{x} \).
   \begin{align*}
   &A. x = -2 \quad B. x = -1 \quad C. x = 0 \quad D. x = 1 \quad E. x = 2
   \end{align*}

Regular problems: show all your work

7. Show that the equation \( x^7 + x^3 + x + 2 = 0 \) has exactly one real root.
   Let \( f(x) = x^7 + x^3 + x + 2 \). Since \( f(0) = 2 > 0 \) and \( f(-1) = -1 < 0 \), by the Intermediate Value Theorem \( f(x) \) has a root between \(-1\) and \(0\).
   Now we have to show that \( f(x) \) cannot have more than 1 real root. Suppose \( f(x) \) has at least two roots, \( a \) and \( b \). Then \( f(a) = f(b) = 0 \), and \( f \) is continuous and differentiable everywhere (since it is a polynomial), therefore by Rolle’s Theorem there is a point \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \). But \( f'(x) = 7x^6 + 3x^2 + 1 > 0 \) for all \( x \). This is a contradiction, therefore \( f \) cannot have two distinct real roots. Therefore it has exactly one real root.

8. Find the linear approximation of the function \( f(x) = \cos(x) \) at \( a = \frac{\pi}{2} \).
   \begin{align*}
   f'(x) &= -\sin(x), \quad f'(\frac{\pi}{2}) = -\sin\left(\frac{\pi}{2}\right) = -1. \quad \text{Thus the slope of the tangent line is } -1.
   f\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) = 0.
   Then an equation of the tangent line is \( y - 0 = -1 \left(x - \frac{\pi}{2}\right) \)
   \end{align*}
   \begin{align*}
   y &= -x + \frac{\pi}{2}
   The linear approximation is \( L(x) = -x + \frac{\pi}{2} \).
9. Find the intervals of increase and decrease of the function \( f(x) = x^4 - 4x^3 + 5 \).

\[
f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)\]

\( f'(x) = 0 \) at \( x = 0 \) and \( x = 3 \). Checking each interval, we have:
- On \((−\infty, 0)\) and on \((0, 3)\) \( f'(x) < 0 \), so \( f(x) \) is decreasing;
- On \((3, +\infty)\) \( f'(x) > 0 \), so \( f(x) \) is increasing.

10. Find the absolute maximum and minimum values of \( f(x) = x\cos(y) + xy^2 - 3y = 0 \) at the point \((0, 0)\).

Differentiating \( x\cos(y(x)) + x(y(x))^2 - 3y(x) = 0 \) implicitly with respect to \( x \) gives:
- \( \cos(y(x)) + x(-\sin(y(x)))y'(x) + 2xy(x)y'(x) - 3y'(x) = 0 \), or
- \( \cos y - x\sin(y(y))y' + y^2 + 2xyy' - 3y' = 0 \).

When \( x = y = 0 \), \( 1 - 0 + 0 - 3y' = 0 \), so \( 3y' = 1 \), and \( y' = \frac{1}{3} \).

11. At noon, ship A is 120 km east of ship B. Ship A is sailing west at 20 km/h and ship B is sailing south at 30 km/h. How fast is the distance between the ships changing at 2:00 PM?

**First draw a diagram:**

<table>
<thead>
<tr>
<th>At noon</th>
<th>At 2:00 PM</th>
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Let \( P \) be the starting point of ship B. Let \( x = |AP|, y = |BP|, \) and \( z = |AB| \). Then \( x^2 + y^2 = z^2 \). Remember that all of these are functions of time:

\[
(x(t))^2 + (y(t))^2 = (z(t))^2.
\]

Differentiating both sides with respect to \( t \) gives

\[
2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t), \quad \text{or, after simplifying,}
\]

\[
x' + yy' = zz'.
\]

At 2 PM, \( x = 120 - 2 \cdot 20 = 80, y = 2 \cdot 30 = 60, \) and \( z = \sqrt{x^2 + y^2} = \sqrt{80^2 + 60^2} = 100 \).

Also, \( x' = -20 \) (it’s negative because the distance \( x \) is decreasing) and \( y' = 30 \). Therefore

\[
80(-20) + 60 \cdot 30 = 100z',
\]

\[
-1600 + 1800 = 100z',
\]

\[
200 = 100z',
\]

\[
z' = 2.
\]

So the distance between the ships is changing at a rate of 2 km/h.

12. Find the absolute maximum and minimum values of \( f(x) = x^4 - 4x^3 + 5 \) on the interval \([-1, 5]\).

**First find the critical numbers of \( f(x) \):**

\[
f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3).
\]

\( f'(x) = 0 \) at \( x = 0 \) and \( x = 3 \). The values of \( f(x) \) at the critical numbers are: \( f(0) = 5 \), \( f(3) = 81 - 108 + 5 = -22 \).

**Now find the values of \( f(x) \) at the endpoints of the interval:**

\[
f(-1) = 1 + 4 + 5 = 10,
\]

\[
f(5) = 625 - 500 + 5 = 130.
\]

The largest of the above values, namely 130, is the absolute maximum value of \( f(x) \) on the given interval, and the smallest of those values, namely –22, is the absolute minimum value.