MATH 75
Test 2 - Solutions
June 16, 2005

Multiple choice questions: circle the correct answer

1. Find the derivative of $f(x) = \sin(4x^2)$.

   A. $\cos(4x^2)$ **B.** $\cos(8x^2)$ **(C.)** $8x\cos(4x^2)$ **D.** $-4x^2\cos(8x)$ **E.** $-\cos(x)(4x^2)$

2. Find the vertical asymptotes of $f(x) = \dfrac{1-x^2}{x^2-4x}$.

   A. $x = 0$ **B.** $x = 4$ **(C.)** $x = 0$ and $x = 4$ **D.** $y = -1$ **E.** $y = -4$

3. Evaluate the limit: $\displaystyle\lim_{x\to-\infty} \frac{x^2+10}{x^3-3}$.

   (A.) 0 **B.** 1 **C.** $-\frac{10}{3}$ **D.** ∞ **E.** $-\infty$

4. If $f(t) = \dfrac{1}{x^2}$, find $f''(-1)$.

   A. -6 **B.** -2 **C.** 0 **D.** 2 **(E.)** 6

5. How many inflection points does the function $y = x + \dfrac{1}{x}$ have?

   (A.) 0 **B.** 1 **C.** 2 **D.** 3 **E.** infinitely many

6. Find the local minimum of $y = x + \dfrac{1}{x}$.

   A. $x = -2$ **B.** $x = -1$ **C.** $x = 0$ **(D.)** $x = 1$ **E.** $x = 2$

Regular problems: show all your work

7. Show that the equation $x^7 + x^3 + x + 2 = 0$ has exactly one real root.

   Let $f(x) = x^7 + x^3 + x + 2$. Since $f(0) = 2 > 0$ and $f(-1) = -1 < 0$, by the Intermediate Value Theorem $f(x)$ has a root between -1 and 0.
   Now we have to show that $f(x)$ cannot have more than 1 real root. Suppose $f(x)$ has at least two roots, a and b. Then $f(a) = f(b) = 0$, and f is continuous and differentiable everywhere (since it is a polynomial), therefore by Rolle's Theorem there is a point c between a and b such that $f'(c) = 0$. But $f'(x) = 7x^6 + 3x^2 + 1 > 0$ for all x. This is a contradiction, therefore f cannot have two distinct real roots. Therefore it has exactly one real root.

8. Find the linear approximation of the function $f(x) = \cos(x)$ at $a = \dfrac{\pi}{2}$.

   $f'(x) = -\sin(x)$, $f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$. *Thus the slope of the tangent line is* -1.
   $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$.
   Then an equation of the tangent line is $y - 0 = -1\left(x - \frac{\pi}{2}\right)$
   $y = -x + \frac{\pi}{2}$
   The linear approximation is $L(x) = -x + \frac{\pi}{2}$

9. Find the intervals of increase and decrease of the function $f(x) = x^4 - 4x^3 + 5$.

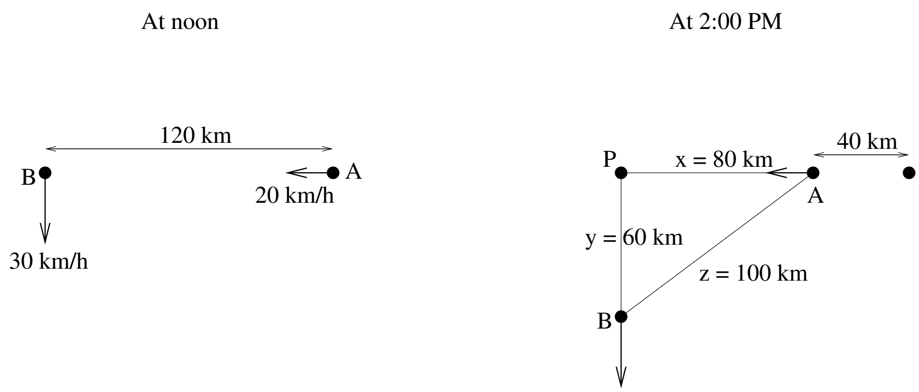
$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$
$f'(x) = 0$ at $x = 0$ and $x = 3$. Checking each interval, we have:
on $(-\infty, 0)$ and on $(0, 3)$ $f'(x) < 0$, so $f(x)$ is decreasing;
on $(3, +\infty)$ $f'(x) > 0$, so $f(x)$ is increasing.

10. Find the slope of the tangent line to the curve $x \cos y + xy^2 - 3y = 0$ at the point $(0, 0)$.

Differentiating $x\cos(y(x)) + x(y(x))^2 - 3y(x) = 0$ implicitly with respect to x gives:
$\cos(y(x)) + x(-\sin(y(x))y'(x) + (y(x))^2 + x2y(x)y'(x) - 3y'(x) = 0$, *or*
$\cos y - x\sin(y)y' + y^2 + 2xyy' - 3y' = 0$.
When $x = y = 0$, $1 - 0 + 0 + 0 - 3y' = 0$, so $3y' = 1$, and $y' = \frac{1}{3}$.

11. At noon, ship A is 120 km east of ship B. Ship A is sailing west at 20 km/h and ship B is sailing south at 30 km/h. How fast is the distance between the ships changing at 2:00 PM?

First draw a diagram:



Let P be the starting point of ship B. Let $x = |AP|$, $y = |BP|$, and $z = |AB|$. Then $x^2 + y^2 = z^2$. Remember that all of these are functions of time:
$(x(t))^2 + (y(t))^2 = (z(t))^2$.
Differentiating both sides with respect to t gives
$2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t)$, *or, after simplifying,*
$xx' + yy' = zz'$.
At 2 PM, $x = 120 - 2 \cdot 20 = 80$, $y = 2 \cdot 30 = 60$, and $z = \sqrt{x^2 + y^2} = \sqrt{80^2 + 60^2} = 100$.
Also, $x' = -20$ (it's negative because the distance x is decreasing) and $y' = 30$. Therefore
$80(-20) + 60 \cdot 30 = 100z'$
$-1600 + 1800 = 100z'$
$200 = 100z'$
$z' = 2$
So the distance between the ships is changing at a rate of 2 km/h.

12. Find the absolute maximum and minimum values of $f(x) = x^4 - 4x^3 + 5$ on the interval $[-1, 5]$.

First find the critical numbers of $f(x)$:
$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. $f'(x) = 0$ at $x = 0$ and $x = 3$. The values of f at the critical numbers are: $f(0) = 5$,
$f(3) = 81 - 108 + 5 = -22$.
Now find the values of f at the endpoints of the interval:
$f(-1) = 1 + 4 + 5 = 10$,
$f(5) = 625 - 500 + 5 = 130$.
The largest of the above values, namely 130, is the absolute maximum value of $f(x)$ on the given interval, and the smallest of those values, namely -22, is the absolute minimum value.