MATH 76
Test 3 - Solutions

December 6, 2004

1. *$r = 0$ when $\sin(6\theta) = 0$. If $\theta = 0$, $\sin(6\theta) = 0$. The smallest positive value of θ when $\sin(6\theta) = 0$ is when $6\theta = \pi$, or $\theta = \frac{\pi}{6}$. So one loop lies between $\theta = 0$ and $\theta = \frac{\pi}{6}$.*

$$A = \frac{1}{2}\int_0^{\pi/6} 4\sin^2(6\theta)d\theta = 2\int_0^{\pi/6}\sin^2(6\theta)d\theta = 2\int_0^{\pi/6}\frac{1}{2}(1-\cos(12\theta))d\theta$$
$$= \int_0^{\pi/6}(1-\cos(12\theta))d\theta = \left(\theta - \frac{\sin(12\theta)}{12}\right)\Bigg|_0^{\pi/6} = \left(\frac{\pi}{6} - 0\right) = \frac{\pi}{6}.$$

2. *Since $r' = 12\cos(6\theta)$, the length of one loop is $\int_0^{\pi/6}\sqrt{4\sin^2(6\theta) + 144\cos^2(6\theta)}d\theta$.*

There are 12 loops, so the length of the whole curve is $L = 12\int_0^{\pi/6}\sqrt{4\sin^2(6\theta) + 144\cos^2(6\theta)}d\theta$.

3. *Complete the squares: $(x^2 - 2x) + 4(y^2 + 4y) = 19$*
$(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 19 + 1 + 16$
$(x-1)^2 + 4(y+2)^2 = 6^2$
$\frac{(x-1)^2}{6^2} + \frac{(y+2)^2}{3^2} = 1$
The center of the ellipse is $(1, -2)$, $a = 6$, $b = 3$, so the vertecies are $(-5, -2)$ (left), $(7, -2)$ (right), $(1, -5)$ (bottom), and $(1, 1)$ (top).

4. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n + 2 + \sin(n)}{n+1} = \lim_{n\to\infty}\frac{3 + \frac{2}{n} + \frac{\sin(n)}{n}}{1 + \frac{1}{n}} = 3$, *so the sequence is convergent.*

5. (a) $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ *is a geometric series with $r = \frac{1}{e}$. Since $|r| < 1$, the series is convergent. Note: the integral test can also be used.*

   (b) *For the series $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$ we use the root test: $\lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{(\ln n)^n}\right|} = \lim_{n\to\infty}\frac{1}{\ln n} = 0 < 1$, so the series is convergent.*

   (c) *We compare the series $\sum_{n=1}^{\infty}\frac{\arctan n}{n}$ with $\sum_{n=1}^{\infty}\frac{\pi/4}{n}$. Since $0 < \frac{\pi/4}{n} < \frac{\arctan n}{n}$ for all positive integers n and $\sum_{n=1}^{\infty}\frac{\pi/4}{n} = \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, the series $\sum_{n=1}^{\infty}\frac{\arctan n}{n}$ is also divergent.*

6. *The ratio test says that the series $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ is convergent when*

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/\sqrt{n+1}}{x^n/\sqrt{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}\sqrt{n}}{x^n\sqrt{n+1}}\right| = \lim_{n\to\infty}\left|x\sqrt{\frac{n}{n+1}}\right| = |x|\lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}} = |x| < 1$$

and divergent when $|x| > 1$. So the radius of convergence is 1.
Check the endpoints: when $x = 1$, $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is divergent, and when $x = -1$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}}$ is convergent by the alternating series test. Therefore the interval of convergence is $[-1, 1)$.

7. *The Maclaurin series is $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \ldots$.*
$f(0) = \ln(1) = 0$,
$f'(x) = \frac{1}{1+x}$, *so* $f'(0) = 1$,
$f''(x) = ((1+x)^{-1})' = -(1+x)^{-2}$, *so* $f''(0) = -1$,
$f'''(x) = 2(1+x)^{-3}$, *so* $f'''(0) = 2$,
$f^{(4)}(x) = -6(1+x)^{-4}$, *so* $f^{(4)}(0) = -6$.
Thus we have $\ln(1+x) = x + \frac{-1}{2}x^2 + \frac{2}{6}x^3 + \frac{-6}{24}x^4 + \ldots = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$.