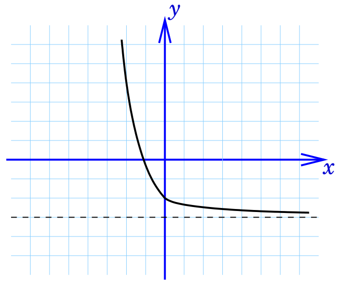
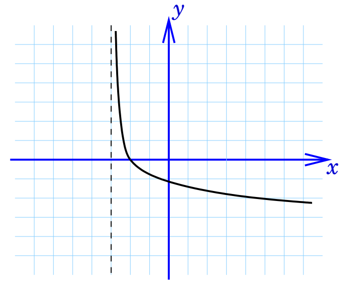
MATH 76
Test 1 - Solutions
September 27, 2004

1. $y = e^{-x} - 3 \;\Rightarrow\; y+3 = e^{-x} \;\Rightarrow\; \ln(y+3) = -x \;\Rightarrow\; x = -\ln(y+3) \;\Rightarrow\; f^{-1}(y) = -\ln(y+3)$
$\Rightarrow\; f^{-1}(x) = -\ln(x+3)$



$y = f(x)$



$y = f^{-1}(x)$

2. (a) $e^{\ln 2 + \ln 3} = e^{\ln 6} = 6$

   (b) $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ *since* $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ *and* $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.

3. *Using L'Hospital's rule,* $\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)} = \lim_{x\to 0} \frac{e^x}{2\cos(2x)} = \frac{1}{2}$

4. *Integrate by parts with* $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$:
$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$

5. $\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^3(x)dx = \int_0^{\frac{\pi}{2}} \sin^2(x)\cos^2(x)\cos(x)dx = \int_0^{\frac{\pi}{2}} \sin^2(x)(1-\sin^2(x))\cos(x)dx =$
(Let $u = \sin(x)$*, then* $du = \cos(x)dx$*, and the new limits of integration are* $\sin(0) = 0$ *and* $\sin\left(\frac{\pi}{2}\right) = 1$*)*
$= \int_0^1 u^2(1-u^2)du = \int_0^1 (u^2 - u^4)du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

6. *Using the trig substitution* $x = 2\sin(t)$, $dx = 2\cos(t)dt$, $\sqrt{4-x^2} = \sqrt{4 - 4\sin^2(t)} = 2\cos(t)$,
$\int \sqrt{4-x^2}dx = \int 2\cos(t)\cdot 2\cos(t)dt = 4\int \cos^2(t)dt = 4\int \frac{1}{2}(\cos(2t)+1)dt$
$= 2\int(\cos(2t)+1)dt = 2\left(\frac{1}{2}\sin(2t) + t\right) + c = \sin(2t) + 2t + c = 2\sin(t)\cos(t) + 2t + c$
$= \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\left(\frac{x}{2}\right) + c$

7. *Step I. Long division gives* $\frac{2x^2+3x+13}{x^2+x-2} = 2 + \frac{x+17}{x^2+x-2}$

   Step II. Factor the denominator: $x^2 + x - 2 = (x-1)(x+2)$

   Step III. Partial fraction decomposition: $\frac{x+17}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$
   $x + 17 = A(x+2) + B(x-1) \;\Rightarrow\; x + 17 = (A+B)x + (2A - B) \;\Rightarrow$
   $A + B = 1$ *and* $2A - B = 17 \;\Rightarrow\; A = 6$ *and* $B = -5$

   Step IV. $\int \frac{2x^2+3x+13}{x^2+x-2}dx = \int 2dx + \int \frac{6}{x-1}dx - \int \frac{5}{x+2}dx$
   $= 2x + 6\ln|x-1| - 5\ln|x+2| + c$