Math 76 **Solutions to practice problems for test 2**

1. Estimate the value of $\int_{-5}^{7} x^2 dx$ using 6 subintervals and

   $a = -5,\ \ b = 7,\ \ n = 6,\ \ \Delta x = \frac{b-a}{n} = 2,$
   $x_0 = -5,\ \ x_1 = -3,\ \ x_2 = -1,\ \ x_3 = 1,\ \ x_4 = 3,\ \ x_5 = 5,\ \ x_6 = 7.$

   (a) the midpoint rule
   Midpoints: -4, -2, 0, 2, 4, and 6.
   $M_6 = \Delta x(f(-4) + f(-2) + f(0) + f(2) + f(4) + f(6)) = 2(16 + 4 + 0 + 4 + 16 + 36) = 152$

   (b) the trapezoidal rule
   $T_6 = \frac{\Delta x}{2}(f(-5) + 2f(-3) + 2f(-1) + 2f(1) + 2f(3) + 2f(5) + f(7)) =$
   $= 1(25 + 18 + 2 + 2 + 18 + 50 + 49) = 164$

   (c) Simpson's rule
   $S_6 = \frac{\Delta x}{3}(f(-5) + 4f(-3) + 2f(-1) + 4f(1) + 2f(3) + 4f(5) + f(7)) =$
   $\frac{2}{3}(25 + 36 + 2 + 4 + 18 + 100 + 49) = \frac{2}{3} \cdot 234 = 156$

2. Evaluate the integrals (if convergent).

   (a) $\int_2^{\infty} e^{-x} dx = \lim_{t\to\infty}\left(\int_2^t e^{-x} dx\right) = \lim_{t\to\infty}\left((-e^{-x})\Big|_2^t\right) = \lim_{t\to\infty}(-e^{-t} + e^{-2}) = 0 + e^{-2} = e^{-2}$

   (b) $\int_{-\infty}^{0} \sin x dx = \lim_{t\to-\infty}\left(\int_t^0 \sin x dx\right) = \lim_{t\to-\infty}\left(-\cos x\Big|_t^0\right) = \lim_{t\to-\infty}(-\cos 0 + \cos t)$
   the limit does not exist, so the integral is divergent.

   (c) $\int_3^5 \frac{1}{x-5} dx$ *(the integrand has a vertical asymptote at 5)*
   $= \lim_{t\to5^-}\int_3^t \frac{1}{x-5}dx = \lim_{t\to5^-}\left((\ln|x-5|)\Big|_3^t\right) = \lim_{t\to5^-}(\ln|t-5| - \ln|-2|) = -\infty - \ln 2 = -\infty$

   (d) $\int_0^{13} \frac{1}{\sqrt{|x-4|}} dx$ *(the integrand has a vertical asymptote at 4)*
   $= \int_0^4 \frac{1}{\sqrt{|x-4|}}dx + \int_4^{13} \frac{1}{\sqrt{|x-4|}}dx =$
   if $0 < x < 4,\ \ x - 4$ *is negative, so* $|x-4| = -x + 4,$ *and*
   if $4 < x < 13,\ \ x - 4$ *is positive, so* $|x-4| = x - 4,$ *thus we have*
   $= \int_0^4 \frac{1}{\sqrt{-x+4}}dx + \int_4^{13} \frac{1}{\sqrt{x-4}}dx$ *We have to evaluate each integral.*
   The first one is $\int_0^4 \frac{1}{\sqrt{-x+4}}dx = \lim_{x\to4^-}\int_0^t \frac{1}{\sqrt{-x+4}}dx$
   substitution: $u = -x + 4,\ \ du = -dx$
   $= \lim_{x\to4^-}\int_4^{-t+4} -\frac{1}{\sqrt{u}}du = -\lim_{x\to4^-}\frac{u^{1/2}}{1/2}\Big|_4^{-t+4} = -\lim_{x\to4^-} 2\sqrt{u}\Big|_4^{-t+4} = -\lim_{x\to4^-}(2\sqrt{-t+4} - 4) = 4$
   The second integral: $\int_4^{13} \frac{1}{\sqrt{x-4}}dx = \lim_{t\to4^+}\int_t^{13} \frac{1}{\sqrt{x-4}}dx$
   substitution: $u = x - 4,\ \ du = dx$
   $= \lim_{x\to4^+}\int_{t-4}^{9} \frac{du}{\sqrt{u}} = \lim_{x\to4^+} 2\sqrt{u}\Big|_{t-4}^{9} = \lim_{x\to4^+}(6 - 2\sqrt{t-4}) = 6$
   Therefore $\int_0^{13} \frac{1}{\sqrt{|x-4|}}dx = 4 + 6 = 10$

3. Find the lenth of the curve:

(a) $y = \ln x, \ \ 1 \le x \le \sqrt{3}$ $\qquad L = \int_1^{\sqrt{3}} \sqrt{1 + \left(\frac{1}{x}\right)^2} dx = \int_1^{\sqrt{3}} \sqrt{\frac{x^2+1}{x^2}} dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x} dx$

trig substitution: $x = \tan t, \ \ dx = \sec^2 t dt, \ \ \sqrt{x^2+1} = \sqrt{\tan^2 t + 1} = \sec t$

change the limits of integration: when $x = 1, \ \ t = \arctan 1 = \frac{\pi}{4}$,

and when $x = \sqrt{3}, \ \ t = \arctan \sqrt{3} = \frac{\pi}{3}$.

$$= \int_{\pi/4}^{\pi/3} \frac{\sec t}{\tan t} \sec^2 t dt = \int_{\pi/4}^{\pi/3} \frac{\sec^3 t}{\tan t} dt = \int_{\pi/4}^{\pi/3} \frac{\sec^3 t \tan t}{\tan^2 t} dt = \int_{\pi/4}^{\pi/3} \frac{\sec^2 t \sec t \tan t}{\sec^2 t - 1} dt =$$

substitution: $u = \sec t, \ \ du = \sec t \tan t dt$.

change the limits of integration: the lower limit is $\sec \frac{\pi}{4} = \sqrt{2}$, *the upper limit is* $\sec \frac{\pi}{3} = 2$.

$$= \int_{\sqrt{2}}^{2} \frac{u^2}{u^2 - 1} du = \int_{\sqrt{2}}^{2} \left(1 + \frac{1}{u^2 - 1}\right) du = \int_{\sqrt{2}}^{2} \left(1 + \frac{1}{(u-1)(u+1)}\right) du =$$

$$= \int_{\sqrt{2}}^{2} \left(1 + \frac{1/2}{u-1} + \frac{-1/2}{u+1}\right) du \ = \ u + \frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| \Big|_{\sqrt{2}}^{2} \ = \ (2 + \frac{1}{2} \ln 1 - \frac{1}{2} \ln 3) -$$

$$(\sqrt{2} + \frac{1}{2} \ln(\sqrt{2} - 1) - \frac{1}{2} \ln(\sqrt{2} + 1)) = 2 - \frac{1}{2} \ln 3 - \sqrt{2} - \frac{1}{2} \ln(\sqrt{2} - 1) + \frac{1}{2} \ln(\sqrt{2} + 1)$$

(b) $x = y^{3/2}, \ \ 4 \le y \le 9$

$L = \int_4^9 \sqrt{1 + \left(\frac{3}{2}\sqrt{y}\right)^2} dy = \int_4^9 \sqrt{1 + \frac{9}{4} y} dy$ $\qquad$ *substitution:* $u = 1 + \frac{9}{4} y, \ \ du = \frac{9}{4} du$

$$= \frac{4}{9} \int_{10}^{85/4} \sqrt{u} du = \frac{4}{9} \frac{u^{3/2}}{3/2} \Big|_{10}^{85/4} = \frac{8}{27} \left(\left(\frac{85}{4}\right)^{3/2} - 10^{3/2} \right)$$

4. Find the area of the surface obtained by rotating

(a) $y = x^3, \ \ 0 \le x \le 2$ about the x-axis,

$S = 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2} dx = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4} dx$ $\quad$ *substitution:* $u = 1 + 9x^4$, $du = 36x^3 dx$

$$= \frac{2}{36} \pi \int_1^{145} \sqrt{u} du = \frac{1}{18} \pi \ \frac{2}{3} u^{3/2} \Big|_1^{145} = \frac{1}{27} \pi (145^{3/2} - 1)$$

(b) $y = 1 - x^2, \ \ 0 \le x \le 1$ about the y-axis,

$S = 2\pi \int_0^1 x \sqrt{1 + (-2x)^2} dx = 2\pi \int_0^1 x \sqrt{1 + 4x^2} dx$ $\qquad$ *substitution:* $u = 1 + 4x^2$, $du = 8x dx$

$$= \frac{2\pi}{8} \int_1^5 \sqrt{u} du = \frac{2\pi}{8} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 = \frac{1}{6} \pi (5^{3/2} - 1)$$

(c) $x = \sqrt{1 - y^2}, \ \ 0 \le y \le 1$ about the x-axis,

$$S = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{-2y}{2\sqrt{1-y^2}}\right)^2} dy = 2\pi \int_0^1 y \sqrt{1 + \frac{y^2}{1-y^2}} dy = 2\pi \int_0^1 y \sqrt{\frac{1}{1-y^2}} dy = 2\pi \int_0^1 \frac{y}{\sqrt{1-y^2}} dy$$

substitution: $u = 1 - y^2$, $du = -2y dy$

$$= 2\pi \left(-\frac{1}{2}\right) \int_1^0 \frac{du}{\sqrt{u}} = -\pi \int_1^0 u^{-1/2} du = -\pi \ \frac{u^{1/2}}{1/2} \Big|_1^0 = -2\pi\sqrt{u} \Big|_1^0 = 2\pi$$

(d) $x = \sqrt{y}, \ \ 1 \le y \le 9$ about the y-axis.

$$S = 2\pi \int_1^9 \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} dy = 2\pi \int_1^9 \sqrt{y} \sqrt{1 + \frac{1}{4y}} dy = 2\pi \int_1^9 \sqrt{y\left(1 + \frac{1}{4y}\right)} dy = 2\pi \int_1^9 \sqrt{y + \frac{1}{4}} dy$$

substitution: $u = y + \frac{1}{4}$, $du = dy$

$$= 2\pi \int_{5/4}^{37/4} \sqrt{u} du = 2\pi \frac{2}{3} u^{3/2} \Big|_{5/4}^{37/4} = \frac{4}{3}\pi \left(\left(\frac{37}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2} \right)$$

5. Find all constants c and k such that $y = ce^{kx}$ is a solution of $y'' + y' - 12y = 0$.

   If $y = ce^{kx}$ *then* $y' = cke^{kx}$ *and* $y'' = ck^2e^{kx}$.
   $ck^2e^{kx} + cke^{kx} - 12ce^{kx} = 0$
   $ce^{kx}(k^2 + k - 12) = 0$
   $ce^{kx}(k+4)(k-3) = 0$
   Since $e^{kx} \neq 0$ *for any* k*, either* $c = 0$ *or* $(k+4)(k-3) = 0$
   So either $c = 0$ *(and* k *is any real number) or* $k = -4$ *or* 3 *(and* c *is any real number).*

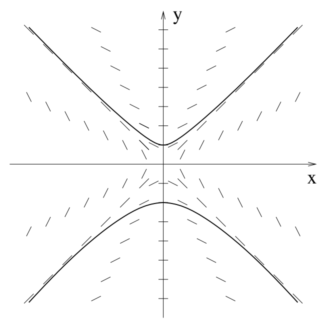
6. Sketch

   (a) a direction field for $y' = \dfrac{x}{y}$,

   Consider $y' = 0$ *(then* $x = 0$*),* $y = 1$ *(*$x = y$*),* $y = -1$ *(*$x = -1$*), etc.*

   (b) solution of $y' = \dfrac{x}{y}$ satisfying $y(0) = 1$,

   (c) solution of $y' = \dfrac{x}{y}$ satisfying $y(0) = -2$.



7. Solve the differential equation

   (a) $y' = \dfrac{x}{y} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x}{y} \quad \Rightarrow \quad ydy = xdx \quad \Rightarrow \quad \int ydy = \int xdx$

   $\Rightarrow \quad \dfrac{y^2}{2} = \dfrac{x^2}{2} + C \quad \Rightarrow \quad y^2 = x^2 + 2C \quad \Rightarrow \quad y^2 = x^2 + K \quad \Rightarrow \quad y = \pm\sqrt{x^2 + K}$

   (b) $y' = \dfrac{xy}{2\ln y} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{xy}{2\ln y} \quad \Rightarrow \quad \dfrac{\ln y dy}{y} = \dfrac{x}{2}dx \quad \Rightarrow \quad \int \dfrac{\ln y dy}{y} = \int \dfrac{x}{2}dx$

   For the left hand side, make the substitution $u = \ln y$*, then* $du = \dfrac{dy}{y}$*,*

   so $\int \dfrac{\ln y dy}{y} = \int u du = \dfrac{u^2}{2} = \dfrac{(\ln y)^2}{2}$

   $\dfrac{(\ln y)^2}{2} = \dfrac{x^2}{4} + C \quad \Rightarrow \quad (\ln y)^2 = \dfrac{x^2}{2} + 2C \quad \Rightarrow \quad (\ln y)^2 = \dfrac{x^2}{2} + K \quad \Rightarrow \quad \ln y = \pm\sqrt{\dfrac{x^2}{2} + K}$

   $\Rightarrow \quad y = e^{\pm\sqrt{\frac{x^2}{2}+K}}$

8. A bacteria cultute starts with 800 bacteria and the growth rate is proportianal to the number of bacteria. After 3 hours the population is 2700. Find the number of bacteria after 5 hours.

   The size of the bacteria culture is given by $y = Ae^{kt}$ *where* A *is the initial population, so we have* $y = 800e^{kt}$*. Since* $y(3) = 2700$*, we have* $800e^{3k} = 2700$

   $e^{3k} = \dfrac{27}{8} \quad \Rightarrow \quad 3k = \ln\left(\dfrac{27}{8}\right) \quad \Rightarrow \quad k = \dfrac{1}{3}\ln\left(\dfrac{27}{8}\right) = \ln\left(\left(\dfrac{27}{8}\right)^{\frac{1}{3}}\right) = \ln\left(\dfrac{3}{2}\right)$

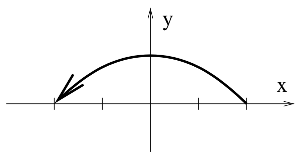
   Then $y(5) = 800e^{5k} = 800\left(e^{k}\right)^5 = 800\left(e^{\ln\left(\frac{3}{2}\right)}\right)^5 = 800\left(\dfrac{3}{2}\right)^5 = \dfrac{800 \cdot 343}{32} = 6075.$

9. Eliminate the parameter to find a Cartesian equation of the curve. Sketh the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

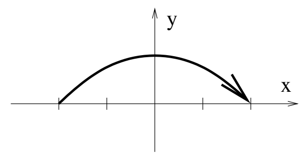
(a) $x = 2\cos\theta$, $y = \sin^2\theta$

$x^2 = 4\cos^2\theta$, *so* $\dfrac{x^2}{4} = \cos^2\theta$. *Therefore* $\dfrac{x^2}{4} + y = 1$, *or* $y = 1 - \dfrac{x^2}{4}$.

The graph is the part of parabola where $-2 \le x \le 2$ (because $-1 \le \cos\theta \le 1$) and $0 \le y \le 1$ (because $0 \le \sin^2\theta \le 1$).



$0 \le \theta \le \pi$

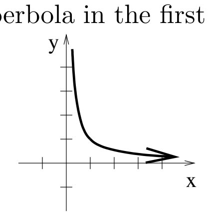


$\pi \le \theta \le 2\pi$

(b) $x = e^t$, $y = e^{-t}$

$xy = 1$ *or* $y = \dfrac{1}{x}$.

The graph is the part of the hyperbola in the first quadrant because e^a is always positive.



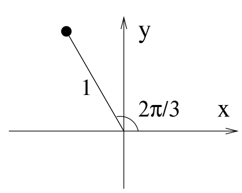
10. Find an equation of the tangent line to the curve $x = \sin t$, $y = \sin(t + \sin t)$ at $(0, 0)$.

The point $(0,0)$ *corresponds to* $t = 0$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos(t+\sin t)(1+\cos t)}{\cos t}. \qquad \textit{If } t = 0,\ \frac{dy}{dx} = \frac{\cos(0+\sin 0)(1+\cos 0)}{\cos 0} = \frac{1\cdot 2}{1} = 2.$$

The slope is 2 *and the tangent line passes through the origin, therefore its equation is* $y = 2x$.

11. (a) Plot the point whose polar coordinates are $\left(1, \dfrac{2\pi}{3}\right)$. Find the Cartesian coordinates of this point.



$x = 1\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$, $y = 1\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$.

(b) Find polar coordinates (with $r > 0$) of the point whose Cartesian coordinates are $(\sqrt{3}, -1)$.

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{(3+1)} = 2$, $\tan\theta = \dfrac{-1}{\sqrt{3}}$, *therefore* $\theta = -\dfrac{\pi}{6}$ *or* $\dfrac{11\pi}{6}$. *So polar coordinates are* $\left(2, -\dfrac{\pi}{6}\right)$ *or* $\left(2, \dfrac{11\pi}{6}\right)$.