Math 76 **Solutions to practice problems for test 3**

1. Find the area of the region enclosed by one loop of $r = \sin(4\theta)$.

   Since $r = 0$ when $4\theta = 0$ and when $4\theta = \pi$, i.e. when $\theta = 0$ and when $\theta = \dfrac{\pi}{4}$, one loop lies in the region $0 \leq \theta \leq \dfrac{\pi}{4}$.

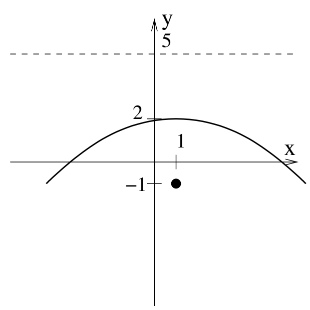
   $$A = \frac{1}{2}\int_0^{\pi/4} \sin^2(4\theta)d\theta = \frac{1}{4}\int_0^{\pi/4}(1-\cos(8\theta))d\theta = \frac{1}{4}\left(\theta - \frac{\sin(8\theta)}{8}\right)\Big|_0^{\pi/4} = \frac{\pi}{16}$$

2. Find the length of the curve given by $r = 5\cos\theta$, $0 \leq \theta \leq \dfrac{3\pi}{4}$.

   $$L = \int_0^{\frac{3\pi}{4}} \sqrt{(5\cos\theta)^2 + (-5\sin\theta)^2}d\theta = \int_0^{\frac{3\pi}{4}} \sqrt{25\cos^2\theta + 25\sin^2\theta}d\theta = \int_0^{\frac{3\pi}{4}} \sqrt{25}d\theta$$
   $$= \int_0^{\frac{3\pi}{4}} 5d\theta = 5 \cdot \frac{3\pi}{4} = \frac{15\pi}{4}.$$

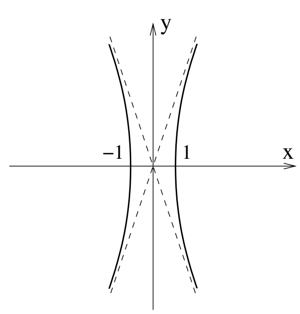
3. Find an equation and sketch the graph of the parabola with focus at $(1, -1)$ and directrix $y = 5$.

   The distance between the focus and the directrix is 6, therefore the distance between the focus and the vertex is 3, so the vertex is at $(1, 2)$. Since the vertex and the focus are below the directrix, the parabola opens downward. So $p = -3$, $a = \frac{1}{4p} = -\frac{1}{12}$. The equation is $y = -\frac{1}{12}(x-1)^2 + 2$.



4. Find the vertices, foci, and asymptotes of the hyperbola give by $9x^2 - y^2 = 9$ and sketch its graph.

   Divide both sides of the given equation by 9: $x^2 - \dfrac{y^2}{9} = 1$ or $\dfrac{x^2}{1^2} - \dfrac{y^2}{3^2} = 1$, so $a = 1$, $b = 3$. Then $c = \sqrt{1^2 + 3^2} = \sqrt{10}$.
   Therefore the vertices are at $(1, 0)$ and $(-1, 0)$, foci are at $(\sqrt{10}, 0)$ and $(-\sqrt{10}, 0)$, and asymptotes are $y = \pm 3x$.



5. Find the vertices and foci of the ellipse given by $9x^2 - 18x + 4y^2 = 27$ and sketch its graph.

Complete the square: $9x^2 - 18x = 9(x-m)^2 + n$

$9x^2 - 18x = 9x^2 - 18mx + 9m^2 + n$

$m = 1$, $n = -0$, *so* $9x^2 - 18x = 9(x-1)^2 - 9$.

Then the given equation can be written as $9(x-1)^2 - 9 + 4y^2 = 27$

$9(x-1)^2 + 4y^2 = 36$

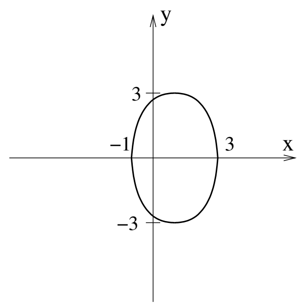
Divide both sides by 36:

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{(x-1)^2}{2^2} + \frac{y^2}{3^2} = 1$$

$a = 2$, $b = 3$, $c = \sqrt{3^2 - 2^2} = \sqrt{5}$.

Therefore the center is at $(1,0)$, *and vertices are at* $(3,0)$, $(-1,0)$, $(1,3)$, $(1,-3)$, *foci are at* $(1,\sqrt{5})$ *and* $(1,-\sqrt{5})$.



6. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1, \text{ so the sequence converges.}$$

(b) $a_n = 2 + \cos(n\pi)$

$$a_n = \begin{cases} 1 & \text{when } n \text{ is odd} \\ 3 & \text{when } n \text{ is even} \end{cases}, \text{ so the sequence does not have a limit.}$$

7. Determine whether the series is convergent or divergent. Explain your reason. If the series is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \arctan n$

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0, \text{ so the series diverges by the test for divergence.}$$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{5}}{3^n}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{5}}{3^n} = \sum_{n=1}^{\infty} \frac{\sqrt{5}}{3} \left(\frac{-1}{3}\right)^{n-1} = \frac{\sqrt{5}}{3} \frac{1}{1-(-\frac{1}{3})} = \frac{\sqrt{5}}{3} \frac{1}{4/3} = \frac{\sqrt{5}}{4}.$$

8. Determine whether the series is convergent or divergent. Explain your reason.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

$\dfrac{\sin^2 n}{n\sqrt{n}} \le \dfrac{1}{n\sqrt{n}}$, *and* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ *converges because it is a p-series with* $p = \dfrac{3}{2} > 1$*, so by the comparison test* $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ *converges.*

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$

This series satistfies all the conditions of the alternating series test: $\dfrac{n}{2^n} > 0$,

$\dfrac{n+1}{2^{n+1}} \le \dfrac{n}{2^n}$ *because multiplying both sides of this by* 2^n *gives* $n+1 \le 2n$ *which is true for all* $n \ge 1$,

and using L'Hospital's rule we have $\displaystyle\lim_{n\to\infty} \frac{n}{2^n} = \lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{\ln(2)2^x} = 0.$

Therefore the series is convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{n+2}{(n+1)!}}{\frac{n+1}{n!}} = \lim_{n\to\infty} \frac{(n+2)n!}{(n+1)(n+1)!} = \lim_{n\to\infty} \frac{n+2}{(n+1)(n+1)}$$

$= \displaystyle\lim_{n\to\infty} \frac{\frac{1}{n} + \frac{2}{n^2}}{(1+\frac{1}{n})(1+\frac{1}{n})} = 0 < 1$, *therefore the series is convergent by the ratio test.*

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n - 1}$

We compare this series with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

Since $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2-5n}{n^3+n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{(n^2-5n)n}{n^3+n-1} = \lim_{n\to\infty} \frac{n^3 - 5n^2}{n^3+n-1} = 1$ *and the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent, the given series is also divergent by the limit comparison test.*

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln^2((n+1)^3)}$

$$\int_1^{\infty} \frac{1}{(x+1)\ln^2((x+1)^3)}dx = \int_1^{\infty} \frac{1}{(x+1)(\ln((x+1)^3))^2}dx = \int_1^{\infty} \frac{1}{(x+1)(3\ln(x+1))^2}dx$$

$$= \int_1^{\infty} \frac{1}{9(x+1)\ln^2(x+1)}dx = \frac{1}{9}\int_1^{\infty} \frac{1}{(x+1)\ln^2(x+1)}dx = \frac{1}{9}\lim_{t\to\infty}\int_1^{t} \frac{1}{(x+1)\ln^2(x+1)}dx$$

Make the substitution $u = \ln(x+1)$, $du = \frac{1}{x+1}dx$, *change the limits of integration: the lower limit becomes* $\ln(2)$*, and the upper limit becomes* $\ln(t+1)$*. Then we have*

$$\frac{1}{9}\lim_{t\to\infty}\int_{\ln(2)}^{\ln(t+1)} \frac{1}{u^2}du = \frac{1}{9}\lim_{t\to\infty}\left(-\frac{1}{u}\Big|_{\ln(2)}^{\ln(t+1)}\right) = \frac{1}{9}\lim_{t\to\infty}\left(-\frac{1}{\ln(t+1)} + \frac{1}{\ln(2)}\right) = \frac{1}{9\ln(2)}.$$

Since the integral is convergent, the given series is convergent by the integral test.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$

$\displaystyle\lim_{n\to\infty} |a|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^n}{3^{1+3n}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{3}\cdot\frac{n^n}{3^{3n}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{3^{\frac{1}{n}}}\cdot\frac{n}{3^3} = \infty$, *therefore the series is divergent by the root test.*

9. Find the radius of convergence and the interval of convergence of the series.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}n^2}{(n+1)^2 x^n}\right| = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2} x\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot \lim_{n\to\infty} |x| = |x|.$

By the ratio test the series is convergent when $|x| < 1$ *and divergent when* $|x| > 1$. *Thus the radius of convergence is* 1.

If $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent.*

If $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *is also convergent, therefore the interval of convergence is* $[-1, 1]$.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}n3^n}{(n+1)3^{n+1}x^n}\right| = \lim_{n\to\infty} \left|\frac{n}{(n+1)3} x\right| = \lim_{n\to\infty} \frac{n}{(n+1)3} \lim_{n\to\infty} |x| = \frac{|x|}{3}.$

By the ratio test the series is convergent when $\frac{|x|}{3} < 1$, *i.e.* $|x| < 3$, *and divergent when* $\frac{|x|}{3} > 1$, *i.e.* $|x| > 3$. *Thus the radius of convergence is* 3.

If $x = 3$, $\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

If $x = -3$, $\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *is convergent, therefore the interval of convergence is* $[-3, 3)$.

10. Find a power series representation for $\frac{x}{4x+1}$ and determine the interval of convergence.

$\frac{x}{4x+1} = x\frac{1}{4x+1} = x\frac{1}{1-(-4x)} = x\sum_{n=0}^{\infty}(-4x)^n = x\sum_{n=0}^{\infty}(-4)^n x^n = \sum_{n=0}^{\infty}(-4)^n x^{n+1}$

$= x - 4x^2 + 16x^3 - 64x^4 + \dots$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-4)^{n+1}x^{n+2}}{(-4)^n x^{n+1}}\right| = \lim_{n\to\infty} |-4x| = \lim_{n\to\infty} |4x|,$

the series is convergent when $|4x| < 1$ *i.e.* $|x| < \frac{1}{4}$, *and divergent when* $|4x| > 1$ *i.e.* $|x| > \frac{1}{4}$.

If $x = \frac{1}{4}$, $\sum_{n=0}^{\infty}(-4)^n \left(\frac{1}{4}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4}$ *is divergent.*

If $x = -\frac{1}{4}$, $\sum_{n=0}^{\infty}(-4)^n \left(-\frac{1}{4}\right)^{n+1} = \sum_{n=0}^{\infty} -\frac{1}{4}$ *is also divergent, so the interval of convergence is* $\left(-\frac{1}{4}, \frac{1}{4}\right)$.

11. Evaluate the integral $\int \frac{1}{1+x^4}\,dx$ as a power series.

$$\int \frac{1}{1+x^4}\,dx = \int \frac{1}{1-(-x^4)}\,dx = \int \sum_{n=0}^{\infty}(-x^4)^n dx = \int \sum_{n=0}^{\infty}(-1)^n x^{4n} dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1} + C$$

12. Find the Taylor series for $f(x) = \frac{1}{x}$ at $a = 1$.

$$f(x) = \frac{1}{x} = \frac{1}{(x-1)+1} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty}(-(x-1))^n = \sum_{n=0}^{\infty}(-1)^n (x-1)^n$$