CSU FRESNO MATH PROBLEM SOLVING

March 21, 2009 Solutions

Topic 1: Equations and inequalities with radicals, exponents, and logs

1. (MH 9-10 2002) Solve: $2^x = \frac{1}{64}$
   (a) $x = 6$
   (b) $x = -6$
   (c) $x = 4$
   (d) none of the above

   Solution. The equation can be rewritten as $2^x = 2^{-6}$. Therefore $x = -6$.

2. (MH 11-12 2000) Solve for x: $3^{\log_3(8x-4)} = 5$
   (a) $\frac{9}{8}$
   (b) $\frac{9}{4}$
   (c) $\frac{8}{5}$
   (d) $\frac{8}{9}$
   (e) None of the above

   Solution. Since $3^{\log_3 a} = a$, the given equation is equivalent to $8x - 4 = 5$. Therefore $x = \frac{9}{8}$.

3. (MH 9-10 2005) Solve for x: $\sqrt{1+\sqrt{2+\sqrt{x}}} = 3$.
   (a) 78
   (b) 3844
   (c) 15
   (d) none of the above

   Solution. Squaring both sides, we get
   $1+\sqrt{2+\sqrt{x}} = 9$
   $\sqrt{2+\sqrt{x}} = 8$
   $2+\sqrt{x} = 64$
   $\sqrt{x} = 62$
   $x = 3844$

4. (MH 11-12 2005) How many real solutions are there to the equation $\sqrt{x^2+1}+\sqrt{x} = 1$?
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

Solution. The function $\sqrt{x}$ is defined only for $x \geq 0$. But if $x > 0$, then $\sqrt{x^2+1}+\sqrt{x} > 1+0 = 1$. So $x = 0$ is the only solution.

5. (MH 11-12 2003) How many roots does the equation $\sqrt{x^2+1}+\sqrt{x^2+2} = 2$ have?
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) None of the above

   Solution. For any real number x, $\sqrt{x^2+1}+\sqrt{x^2+2} \geq 1+\sqrt{2} > 2$, so the equation has no roots.

6. (MH 9-10 1998) Solve for x: $(6^{x+3} \cdot 6^{2x-1}) = 1$.
   (a) $\frac{2}{3}$
   (b) $\frac{3}{2}$
   (c) $\frac{-2}{3}$
   (d) none of the above

   Solution. The equation can be simplified to $6^{3x+2} = 1$. Then $3x+2 = 0$, so $x = \frac{-2}{3}$.

7. (MH 9-10 2005) Solve for x: $4^x - 4^{x-1} = 12$.
   (a) 2
   (b) 3
   (c) 9
   (d) none of the above

   Solution. The equation can be rewritten as
   $4 \cdot 4^{x-1} - 4^{x-1} = 12$
   $3 \cdot 4^{x-1} = 12$
   $4^{x-1} = 4$
   $x - 1 = 1$
   $x = 2$

8. (MH 11-12 2008) Solve for x: $9^x - 4 \cdot 3^{x+1} + 27 = 0$
   (a) $x = 3$ and $x = 9$
   (b) $x = -1$ and $x = -2$
   (c) $x = 1$ and $x = 2$
   (d) $x = -3$ and $x = -9$

   Solution. The equation can be rewritten as
   $(3^2)^x - 4 \cdot 3 \cdot 3^x + 27 = 0$
   $(3^x)^2 - 12 \cdot 3^x + 27 = 0.$
   Let $3^x = y$, then we get
   $y^2 - 12y + 27 = 0$

$(y-3)(y-9)=0$
This equation has two roots: $y=3$ and $y=9$.
If $y=3$, then $3^x=3$ gives $x=1$. If $y=9$, then $3^x=9$ gives $x=2$.
So the original equation has two roots: $x=1$ and $x=2$.

9. (LF 9-12 2000) The real solution to the equation $\dfrac{81^{x+2}}{9^{3x+4}}=9^{5x+1}$ is $x=$

   (a) $\frac{-1}{3}$
   (b) $\frac{-2}{3}$
   (c) $\frac{-3}{4}$
   (d) $\frac{-1}{6}$
   (e) None of these

   Solution. Let's simplify the left hand side:
   $\dfrac{(9^2)^{x+2}}{9^{3x+4}}=9^{5x+1}$
   $\dfrac{9^{2x+4}}{9^{3x+4}}=9^{5x+1}$
   $9^{-x}=9^{5x+1}$
   $-x=5x+1$
   $6x=-1$
   $x=\frac{-1}{6}$

10. (MH 11-12 2005) Solve for x: $3(8^x)+9(4^x)-30(2^x)=0$.

   (a) 0
   (b) 1
   (c) 2
   (d) -5
   (e) There is no solution

   Solution. The equation can be rewritten as
   $3((2^3)^x)+9((2^2)^x)-30(2^x)=0$
   $3((2^x)^3)+9((2^x)^2)-30(2^x)=0$
   Let $2^x=y$, then we have
   $3y^3+9y^2-30y=0$
   $3y(y^2+3y-10)=0$
   $3y(y-2)(y+5)=0$
   We get three solutions: $y=0$, $y=2$, and $y=-5$.
   The equations $2^x=0$ and $2^x=-5$ have no roots, and $2^x=2$ has one root $x=1$.

11. (MH 11-12 2003) Given that $9^x+9^{-x}=34$, find 3^x+3^{-x}.

   (a) 3
   (b) 6
   (c) 9

(d) 27

(e) 81

Solution. Since $(3^x+3^{-x})^2 = (3^x)^2 + 2\cdot 3^x 3^{-x} + (3^{-x})^2 = 9^x + 2 + 9^{-x} = 34 + 2 = 36$,
$3^x + 3^{-x} = 6$.
Note: the value -6 is impossible because $3^x + 3^{-x} > 0$.

12. (MH 11-12 1997) If $5^{3\log_5 x} = 64$, then

(a) $x = 5$

(b) $x = 125$

(c) $x = \frac{64}{3}$

(d) $x = 4$

(e) None of the above

Solution. The equation can be rewritten as
$(5^{\log_5 x})^3 = 64$
$5^{\log_5 x} = 4$
$x = 4$.

13. (MH 11-12 2005) Find the value of n if $\log_2(\log_5(\log_4 2^n)) = 2$.

(a) 0

(b) 4

(c) 25

(d) 625

(e) 1,250

Solution. Raising 2 to both sides of this equation and simplifying, we get
$2^{\log_2(\log_5(\log_4 2^n))} = 2^2$
$\log_5(\log_4 2^n) = 4$
$5^{\log_5(\log_4 2^n)} = 5^4$
$\log_4 2^n = 625$
$4^{\log_4 2^n} = 4^{625}$
$2^n = 4^{625}$
$2^n = (2^2)^{625}$
$2^n = 2^{1,250}$
$n = 1,250$

14. (MH 11-12 2003) Find the natural n such that $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_n(n+1) = 10$

(a) 9

(b) 10

(c) 100

(d) 1023

(e) Does not exist

Solution. Using the change of base formula $\log_a b = \frac{\log_c b}{\log_c a}$, the given equation can be rewritten as
$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_n(n+1) = 10$
$\frac{\ln 3}{\ln 2} \cdot \frac{\ln 4}{\ln 3} \cdot \frac{\ln 5}{\ln 4} \cdot \ldots \cdot \frac{\ln(n+1)}{\ln n} = 10$
$\frac{\ln(n+1)}{\ln 2} = 10$
$\ln(n+1) = 10 \ln 2$
$\ln(n+1) = \ln 2^{10}$
$n+1 = 2^{10}$
$n = 2^{10} - 1$
$n = 1023$

15. (MH 9-10 1998) Solve for x: $\log_{10}(x^2+3x) + \log_{10}(5x) = 1 + \log_{10}(2x)$.
   (a) 10
   (b) 1
   (c) -5
   (d) $\frac{1}{5}$

Solution. Using $\log_a a = 1$ and $\log_a b + \log_a c = \log_a(bc)$, the given equation can be rewritten as
$\log_{10}(x^2+3x) + \log_{10}(5x) = \log_{10} 10 + \log_{10}(2x)$
$\log_{10}((x^2+3x)(5x)) = \log_{10}(10 \cdot 2x)$
$(x^2+3x)(5x) = 20x$
$5x(x^2+3x) - 20x = 0$
$5x(x^2+3x-4) = 0$
$5x(x+4)(x-1) = 0.$
This equation has three roots: $x=0$, $x=-4$, and $x=1$. The first two values are not roots of the original equation because the logarithmic function is defined only at positive values. So the only solution is $x=1$.

16. (MH 11-12 2006) Solve for x: $\log_2 x + \log_3 x = 3 + \log_2 3 + \log_3 4$
   (a) $\frac{1}{6}$
   (b) $\frac{2}{3}$
   (c) $\frac{3}{2}$
   (d) 6
   (e) 12

Solution. Using the base of change formula, the equation can be rewritten as
$\frac{\ln x}{\ln 2} + \frac{\ln x}{\ln 3} = 3 + \frac{\ln 3}{\ln 2} + \frac{\ln 4}{\ln 3}$
$\frac{\ln x \ln 3 + \ln x \ln 2}{\ln 2 \ln 3} = \frac{3 \ln 2 \ln 3 + (\ln 3)^2 + \ln 4 \ln 2}{\ln 2 \ln 3}$
$\ln x \ln 3 + \ln x \ln 2 = 3 \ln 2 \ln 3 + (\ln 3)^2 + \ln 4 \ln 2$
$\ln x(\ln 3 + \ln 2) = 3 \ln 2 \ln 3 + (\ln 3)^2 + \ln(2^2) \ln 2$
$\ln x(\ln 3 + \ln 2) = 3 \ln 2 \ln 3 + (\ln 3)^2 + 2(\ln 2)^2$
$\ln x = \frac{3 \ln 2 \ln 3 + (\ln 3)^2 + 2(\ln 2)^2}{\ln 3 + \ln 2}$
$\ln x = \frac{(2 \ln 2 + \ln 3)(\ln 2 + \ln 3)}{\ln 3 + \ln 2}$
$\ln x = 2 \ln 2 + \ln 3$
$\ln x = \ln 4 + \ln 3$

$\ln x = \ln 12$
$x = 12$

17. (MH 11-12 2005) Solve $x - xe^{3x-8} = 0$.
    (a) $x = 0$
    (b) $x = \frac{8}{3}$
    (c) $x = -\frac{8}{3}$
    (d) $x = \frac{3}{8}$
    (e) $x = 0$ and $x = \frac{8}{3}$

    Solution. Factor the left hand side:
    $x(1 - e^{3x-8}) = 0$.
    Either $x = 0$ or $1 - e^{3x-8} = 0$. In the second case, $e^{3x-8} = 1$, then $3x - 8 = 0$, so $x = \frac{8}{3}$.
    So we have two solutions: $x = 0$ and $x = \frac{8}{3}$.

18. (MH 11-12 2003) Solve for x: $\sqrt{x^2 - x - 12} < x$
    (a) $x \in (-12, +\infty)$
    (b) $x \in [4, +\infty)$
    (c) $x \in (12, +\infty)$
    (d) No solutions exist
    (e) None of the above

    Solution. Squaring both sides (and remembering that both x and $x^2 - x - 12$ must be nonnegative) we get
    $x^2 - x - 12 < x^2$
    $-x - 12 < 0$
    $x > -12$.
    However, we also need $x \geq 0$ and $x^2 - x - 12 \geq 0$. The latter implies $(x-4)(x+3) \geq 0$, so the solution set is the intersection of:
    $[-12, +\infty)$, $[0, +\infty)$, and $(-\infty, -3] \cup [4, +\infty)$. The anser is $x \in [4, +\infty)$.

19. (MH 11-12 2003) Solve for x: $\log_{x^2-3} 729 > 3$
    (a) $x \in (0, +\infty)$
    (b) $x \in (-\sqrt{12}, -2)$
    (c) $x \in (3, +\infty)$
    (d) $x \in (2, \sqrt{12})$
    (e) (b) or (d)

    Solution. First we note that $x^2 - 3 > 0$.
    Case I: $x^2 - 3 > 1$, i.e. $x^2 > 4$, i.e. $x > 2$. Then the given inequality is equivalent to
    $729 > (x^2 - 3)^3$
    $9 > x^2 - 3$
    $x^2 < 12$.
    So in this case we get $x \in (2, \sqrt{12})$.

Case II: $0 < x^2 - 3 < 1$, i.e. $3 < x^2 < 4$. Then the given inequality is equivalent to
$729 < (x^2 - 3)^3$
$9 < x^2 - 3$
$x^2 > 12$. But the system $3 < x^2 < 4$, $x^2 > 12$ has no solutions.
So the answer is $x \in (2, \sqrt{12})$.

Topic 2: Complex numbers

Simplifying/evaluating expressions involving complex numbers

1. (MH 11-12 2005) Divide $\frac{3-2i}{2+4i}$.
   (a) $-\frac{1}{10} - \frac{2}{5}i$
   (b) $-\frac{1}{10} - \frac{4}{5}i$
   (c) $\frac{7}{10} + \frac{2}{5}i$
   (d) $\frac{4}{5} - \frac{1}{10}i$
   (e) None of the above

   Solution. Multiplying the numerator and the denominator by the conjugate of the denominator and simplifying, we get
   $\frac{3-2i}{2+4i} = \frac{(3-2i)(2-4i)}{(2+4i)(2-4i)} = \frac{6-12i-4i-8}{4+16} = \frac{-2-16i}{20} = -\frac{1}{10} - \frac{4}{5}i.$

2. (MH 11-12 2005) If i is the imaginary number, what is i^{85}?
   (a) 1
   (b) -1
   (c) i
   (d) $-i$
   (e) None of the above

   Solution. Since $i^2 = -1$, we have $i^{85} = i^{84} \cdot i = (i^2)^{42} \cdot i = (-1)^{42} \cdot i = 1 \cdot i = i$.

3. (MH 9-10 2005) Simplify $(1 - (-i)^{318})^2$.
   (a) 4
   (b) i
   (c) 0
   (d) none of the above

   Solution. Since $i^2 = -1$, we have $(1 - (-i)^{318})^2 = (1 - i^{318})^2 = (1 - (i^2)^{159})^2 = (1 - (-1)^{159})^2 = (1 - (-1))^2 = 2^2 = 4$.

4. (MH 11-12 2005) Determine the real part of $(1 + 2i)^5$.
   (a) 1
   (b) 41

(c) 17

(d) 121

(e) None of the above.

Solution. Using the binomial theorem, we have $(1+2i)^5 = 1+5\cdot 2i+10(2i)^2+10(2i)^3+5(2i)^4+(2i)^5 = 1+10i-40-80i+80+32i = 41-38i$. The real part is 41.

5. (MH 9-10 2002) Find: $\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^3$

(a) i

(b) $-i$

(c) -1

(d) 1

Solution 1. $\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^3 = \left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2\cdot\frac{\sqrt{3}i}{2} + 3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}i}{2}\right)^2 + \left(\frac{\sqrt{3}i}{2}\right)^3 = -\frac{1}{8}+\frac{3\sqrt{3}i}{8}+\frac{3\cdot 3}{8}-\frac{3\sqrt{3}i}{8} = \frac{-1+3\sqrt{3}i+9-3\sqrt{3}i}{8} = \frac{8}{8} = 1.$
Note. The above solution is the most straighforward, but it takes time to expand the expression. The solution given below is faster, but requires knowlege of a polar representation.
Solution 2. The polar representation of the number $-\frac{1}{2}+\frac{\sqrt{3}i}{2} = \cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)$ is $e^{\frac{2\pi}{3}i}$, and its cube is $(e^{\frac{2\pi}{3}i})^3 = e^{2\pi i} = \cos(2\pi)+i\sin(2\pi) = 1$.

6. (MH 11-12 2005) Determine the polar representation of $(\sqrt{3}-i)^4$.

(a) $16e^{i\frac{2\pi}{3}}$

(b) $16e^{i\frac{4\pi}{3}}$

(c) $16e^{i\frac{5\pi}{3}}$

(d) 16

(e) None of the above

Solution. Since $\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = 2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right) = 2e^{-\frac{\pi}{6}i}$,
$(\sqrt{3}-i)^4 = \left(2e^{-\frac{\pi}{6}i}\right)^4 = 16e^{-\frac{2\pi}{3}i} = 16e^{i\frac{4\pi}{3}}$.

7. (MH 11-12 2000) Simplify: $\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{10}$

(a) i

(b) $-i$

(c) 1

(d) -1

(e) None of the above

Solution. Using the polar representation, we have
$\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{10} = \left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)^{10} = \cos\left(\frac{10\pi}{4}\right)+i\sin\left(\frac{10\pi}{4}\right) = \cos\left(\frac{5\pi}{2}\right)+i\sin\left(\frac{5\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right) = i.$

8. (MH 11-12 2005) Determine the polar representation of $\frac{2-2i}{1+i}$.

   (a) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

   (b) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

   (c) $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

   (d) $2\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$

   (e) None of the above

   Solution. Let's divide first, and then convert to the polar representation: $\frac{2-2i}{1+i}=\frac{(2-2i)(1-i)}{(1+i)(1-i)}=\frac{2-2i-2i-2}{1+1}=\frac{-4i}{2}=-2i=2(-i)=2\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$.

9. (MH 11-12 2000) Convert to polar notation and multiply: $(1+i)(\sqrt{3}-i)$

   (a) $2\sqrt{3}(\cos 60^0+i\sin 60^0)$

   (b) $2\sqrt{2}(\cos 15^0+i\sin 15^0)$

   (c) $2\sqrt{3}(\cos 30^0+i\sin 30^0)$

   (d) $2\sqrt{2}(\cos 45^0+i\sin 45^0)$

   (e) None of the above

   Solution. Just following the directions... $(1+i)(\sqrt{3}-i)=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\cdot 2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=$

   $\sqrt{2}\left(\cos 45^0+i\sin 45^0\right)\cdot 2\left(\cos(-30^0)+i\sin(-30^0)\right)=2\sqrt{2}\left(\cos 15^0+i\sin 15^0\right)$.

10. (LF 9-12 2000) Suppose $z=1-i$. The real part of $1+z+z^2+z^3+\ldots+z^{99}$ is

   (a) 0

   (b) -1

   (c) $-1-2^{50}$

   (d) $-2^{49}\sqrt{2}$

   (e) None of these

   Solution. Using the formula for the sum of a geometric series, we have
   $1+z+z^2+z^3+\ldots+z^{99}=\frac{1-z^{100}}{1-z}=\frac{1-(1-i)^{100}}{1-(1-i)}=\frac{1-((1-i)^2)^{50}}{i}=\frac{1-(-2i)^{50}}{i}=\frac{1-((2i)^2)^{25}}{i}=\frac{(1-(-4)^{25})i}{-1}=-(1-(-4)^{25})i$. This number has real part 0 because $1-(-4)^{25}$ is real.

11. (MH 11-12 2005) Let $z=x+iy$. Determine the real part of $z^2/\bar{z}$.

   (a) $\frac{x^2-y^2}{x^2+y^2}$

   (b) $\frac{3x^2y+y^3}{x^2+y^2}$

   (c) $\frac{3x^2y-y^3}{x^2+y^2}$

   (d) $\frac{x^3+3xy^2}{x^2+y^2}$

   (e) $\frac{x^3-3xy^2}{x^2+y^2}$

Solution. Let's rewrite the expression in terms of x and y:
$\frac{z^2}{\bar{z}} = \frac{(x+iy)^2}{x-iy} = \frac{(x+iy)^3}{(x-iy)(x+iy)} = \frac{x^3+3x^2yi-3xy^2-y^3i}{x^2+y^2} = \frac{(x^3-3xy^2)+(3x^2y-y^3)i}{x^2+y^2}$. The real part is $\frac{x^3-3xy^2}{x^2+y^2}$.

12. (LF 9-12 2002) Suppose w and z are two complex numbers that satisfy $wz = 1$ and $w+z = -1$. Then $w^{16}+z^{16} =$

    (a) i

    (b) 1

    (c) -1

    (d) $-i$

    (e) None of these

    Solution 1. Solving $w+z=-1$ for w and substituting into the other equation, we get:
    $w = -1-z$
    $(-1-z)z = 1$
    $-z-z^2 = 1$
    $z^2+z+1 = 0$

    $z = \frac{-1\pm\sqrt{1-4}}{2}$

    $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$

    If $z = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$, then $w = -\frac{1}{2}-\frac{\sqrt{3}}{2}i.$
    Note: if $z = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$, then $w = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$, so z and w are switched and the value of $w^{16}+z^{16}$ is the same; so let's consider the case $z = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $w = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$.
    The polar representations are:
    $z = \cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right) = e^{\frac{2\pi}{3}i}$ and $w = \cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right) = e^{-\frac{2\pi}{3}i}$.
    Then $w^{16}+z^{16} = \left(e^{-\frac{2\pi}{3}i}\right)^{16}+\left(e^{\frac{2\pi}{3}i}\right)^{16} = e^{-\frac{32\pi}{3}i}+e^{\frac{32\pi}{3}i} = e^{-\frac{2\pi}{3}i}+e^{\frac{2\pi}{3}i} = w+z = -1.$
    Solution 2. The numbers w and z are roots of the equation $x^2+x+1=0$, therefore they are also roots of $x^3-1=0$. Thus $w^3=1$ and $z^3=1$. Then $w^{16}+z^{16} = w^{15}\cdot w+z^{15}\cdot z = (w^3)^5\cdot w+(z^3)^5\cdot z = w+z = -1.$

Roots of polynomials

Theorem. If $a+bi$ is a root of a polynomial with real coefficients, then $a-bi$ is also a root of this polynomial.
Corollary. A polynomial with real coefficients has an even number of nonreal complex roots.

13. (MH 11-12 1997) If a polynomial with real coefficients has $2+i\sqrt{5}$ and 6 as roots, then another root of the polynomial is:

    (a) $-2+i\sqrt{5}$

    (b) $6i$

    (c) $-2-i\sqrt{5}$

    (d) There need not be another root.

    (e) There is another root but it is none of the above.

Solution. If $2+i\sqrt{5}$ is a root of a polynomial with real coefficients, then its conjugate, $2-i\sqrt{5}$, must also be a root.

14. (MH 11-12 2005) How many roots does the polynomial z^3+64 have?
   (a) No roots
   (b) One real repeated root
   (c) Two real roots, one of which is repeated
   (d) Two real roots and one complex root
   (e) One real root and a pair of complex conjugate roots

   Solution. The polynomial can be factored as $z^3+64 = z^3+4^3 = (z+4)(z^2-4z+16)$. So -4 is one real root. Using quadratic formula, it can be checked that the roots of $(z^2-4z+16)$ are complex conjugate numbers, so the original polynomial has one real root and a pair of complex conjugate roots.

15. (MH 11-12 2000) What is the polynomial of lowest degree with rational coefficients that has $2+\sqrt{3}$ and $1-i$ as some of its roots?
   (a) $x^4-8x^3+12x^2-10x+2$
   (b) $x^4-6x^3+11x^2-10x+2$
   (c) $x^4+6x^3+11x^2+10x+4$
   (d) $x^4-12x^3+11x^2-10x+12$
   (e) None of the above

   Solution. If $1-i$ is a root of a polynomial with real coefficients, then its complex conjugate, $1+i$, is also a root. Similarly, if $2+\sqrt{3}$ is a root of a polynomial with rational coefficients, then its "irrational conjugate" $2-\sqrt{3}$ is also a root. So a polynomial of lowest degree is
   $(x-(1-i))(x-(1+i))(x-(2+\sqrt{3}))(x-(2-\sqrt{3})) = (x-1+i)(x-1-i)(x-2-\sqrt{3})(x-2+\sqrt{3}) = ((x-1)+i)((x-1)-i)((x-2)-\sqrt{3})((x-2)+\sqrt{3}) = ((x-1)^2-i^2)((x-2)^2-(\sqrt{3})^2) = ((x^2-2x+1)+1)((x^2-4x+4)-3) = (x^2-2x+2)(x^2-4x+1) = x^4-6x^3+11x^2-10x+2.$

16. (LF 9-12 1998) The sum of the four distinct complex roots to the polynomial $x^4+2x^3+3x^2+4x+5$ is
   (a) 4
   (b) $\sqrt{5}$
   (c) i
   (d) $4i$
   (e) None of these

   Solution. Let r_1, r_2, r_3, and r_4 be the roots. When a product $(x-r_1)(x-r_2)(x-r_3)(x-r_4)$ is expanded, we get
   $(x-r_1)(x-r_2)(x-r_3)(x-r_4) = x^4-(r_1+r_2+r_3+r_4)x^3+Ax^2+Bx+C$, where A, B, and C are polynomials in r_1, r_2, r_3, and r_4 (in this problem these expressions are irrelevant). So $-(r_1+r_2+r_3+r_4) = 2$, thus the sum of the four roots is -2.