CSU Fresno Problem Solving Session

Problem Solving Sessions website:
`http://zimmer.csufresno.edu/~mnogin/mfd-prep.html`

Math Field Day date: Saturday, April 21, 2012

Math Field Day website:
`http://www.csufresno.edu/math/news_and_events/field_day/`

Geometry, 17 March 2012

Important concepts and theorems

Perimeter, circumference, area, surface area, volume of basic shapes (rectangle, triangle, trapezoid, parallelogram, circle, arc of a circle, sector, prism, pyramid, cone, sphere),

Similar figures have proportional sides; perimeter grows proportionally to the length, area grows proportionally to the square of the length, volume grows proportionally to the cube of the length

Pythagorean theorem

Sum of angles in any triangle is $180°$, in any n-gon $(n-2)\cdot 180°$

Ratios of lengths of sides of $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles

Trigonometric functions: definition (using triangle and unit circle), exact values for some angles

Law of sines, law of cosines

Problems

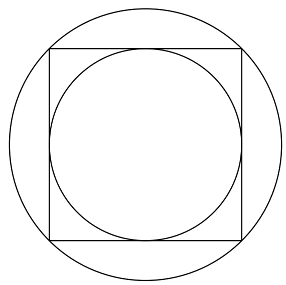
1. (MH 11-12, 2011) A circle has center O and points N and P are on the circle. Suppose that $m\angle NOP$ is $120°$. What is the $m\angle ONP$?
   (a) $30°$
   (b) $35°$
   (c) $60°$
   (d) $70°$
   (e) $125°$

   Solution. Since the points N and P are on the circle and O is the center of the circle, $\overline{NO}$ and $\overline{PO}$ are radii. Thus, they have the same length, making $\triangle NOP$ isoceles. We know that the base angles of any isoceles triangle are equal, so $m\angle ONP = m\angle OPN$. Hence, we have

$$
\begin{aligned}
180^\circ &= m\angle NOP + m\angle ONP + m\angle OPN \\
180^\circ &= 120^\circ + 2(m\angle ONP) \\
60^\circ &= 2(m\angle ONP) \\
30^\circ &= m\angle ONP
\end{aligned}
$$

Answer. (a)

2. (MH 11-12, 2010) A square is drawn inside a circle such that each of its vertices is a point on the circle. Another circle is inscribed in the square. Find the ratio of the area of the circumscribed circle to the area of the inscribed circle.
   (a) $1 : 2$
   (b) $2 : 1$
   (c) $1 : \sqrt{2}$
   (d) $\sqrt{2} : 1$
   (e) None of the above.



Solution. Let x be the side length of the square. Then the diameter of the larger circle is $x\sqrt{2}$, and its radius is $\frac{1}{2}x\sqrt{2}$. The diameter of the smaller circle is x, and its radius is $\frac{1}{2}x$. Therefore, their respective areas are $\frac{1}{2}\pi x^2$ and $\frac{1}{4}\pi x^2$. Thus, the ratio of their areas is $2 : 1$.
Answer. (b)

3. (MH 11-12, 2010) What is the perimeter of a rhombus with diagonals measuring 32 in and 24 in?
   (a) 20 in
   (b) 48 in
   (c) 80 in
   (d) 96 in
   (e) It cannot be determined.

Solution. Recall that the diagonals of a rhombus are perpendicular bisectors of each other. Therefore, let us form a right triangle with side lengths of 12 in and 16 in, and denote x to be the hypotenuse and the side of the rhombus. Now, by the Pythagorean Theorem, we have

$$
\begin{aligned}
12^2 + 16^2 &= x^2 \\
144 + 256 &= x^2 \\
400 &= x^2 \\
20 &= x
\end{aligned}
$$

Also, recall that the lengths of the sides of a rhombus are equal. Thus, $4x = 4(20) = 80$ in is the perimeter of the rhombus.
Answer. (c)

4. (MH 11-12, 2010) A cube and a 3 x 8 x 9 rectangular box have the same volume. What is the area of a side of the cube?

   (a) 3

   (b) 6

   (c) 8

   (d) 9

   (e) 36

   Solution. The volume of the rectangle is $(3)(8)(9) = 216$, which is equal to the volume of the cube. Hence, the length of an edge of the cube is $\sqrt[3]{216} = \sqrt[3]{6^3} = 6$. Thus, the area of one side of the cube is $6^2 = 36$.
   Answer. (e)

5. (MH 11-12, 2010) A square has sides of length 10, and a circle centered at one of its vertices has radius 10. What is the area of the union of the regions enclosed by the square and the circle?

   (a) $100 + 100\pi$

   (b) $100 + 75\pi$

   (c) $100 + 50\pi$

   (d) $100 + 25\pi$

   (e) 100

   Solution. First, let us calculate the area of the part of the circle outside the square:

   $$\text{Area of } \tfrac{3}{4} \text{ circle} = \tfrac{270°}{360°}\pi(10^2) = \tfrac{3}{4}(100\pi) = 75\pi.$$

   Now, we compute the area of the square:

   $$\text{Area of square} = (10^2) = 100.$$

   Then, we add the areas and we have the area of the union:

   $$\text{Area of } \tfrac{3}{4} \text{ circle} + \text{Area of the square} = 75\pi + 100.$$

   Answer. (b)

6. (MH 11-12) Parallelogram $ABCD$ is such that $|AB| = 17$ cm. There are points E on AB and F on CD such that $|EF| = 10$ cm and EF is perpendicular to AB. Find the area of $ABCD$.

   (a) 85 cm^2

   (b) 167 cm^2

   (c) 170 cm^2

   (d) 340 cm^2

   Solution. Recall that the area of a parallelogram is base $\times$ height. We are given

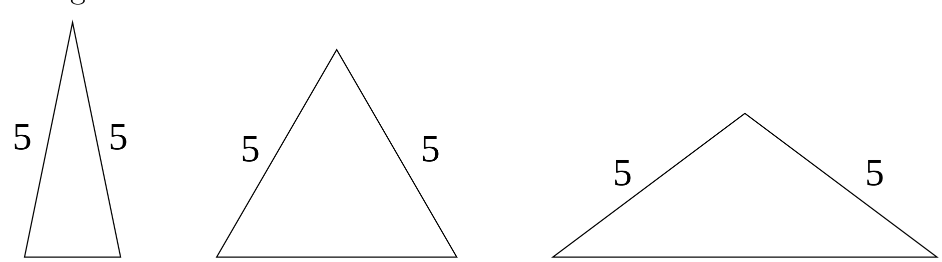
that in parallelogram $ABCD$, $|AB| = 17$ cm. Since any of the four sides can be a base, we will make AB our base. It is also given that line EF is perpendicular to AB, which also makes it perpendicular to CD since CD is parallel to AB, so EF the height and since $|EF| = 10$ cm, the area of the parallelogram is 10 cm $\times$ 17 cm $=$ 170 cm^2.
Answer. (c)

7. (MH 11-12) Which of the following triples of numbers could represent the lengths of the sides of a triangle?
   (a) 2, 2, 5
   (b) 3, 3, 5
   (c) 4, 4, 8
   (d) 5, 5, 15
   (e) None of the above.

   Solution. Recall that in any triangle, the lengths of the two smaller sides add up to be strictly greater than the length of the longest side. Observe that 3, 3, 5 is the only case when this is true.
   Answer. (b)

8. (MH 9-10) An isosceles triangle has equal sides of length 5. Determine the length of the third side.
   (a) 4
   (b) 3
   (c) 6
   (d) It cannot be determined from the information given

   Solution. Based on the given information we cannot determine the length of the third side. More information is needed such as the measurement of one of the internal angles.



   Answer. (d)

9. (MH 9-10) The perimeter of rhombus $RSTU$ is 52 and the diagonal $|RT| = 24$. What is the area of the rhombus?
   (a) 312
   (b) 240
   (c) 120
   (d) 78

Solution. A rhombus has four equal sides. Thus each side of the rhombus is $\frac{52}{4} = 13$. Let X be the center of the rhombus. Since $|RT| = 24$, $|RX| = 12$. Consider the right triangle RXS and apply the Pythagorean theorem to find $|XS|$.
$|RX|^2 + |XS|^2 = |RS|^2$
$12^2 + |XS|^2 = 13^2$
$144 + |XS|^2 = 169$
$|XS|^2 = 25$
$|XS| = 5$
Now calculate the area of the right triangle RXS and multiply it by 4 to get the area of the rhombus:
Area $= 4((\text{base} \times \text{height})/2) = 4 \times (12 \times 5)/2 = 4 \times 30 = 120$.
Answer. (c)

10. (MH 11-12, 2009) Start with an equilateral triangle of side 1. Form a new triangle by joining the midpoints of the sides. Then form a third triangle by joining the midpoints of the second triangle. Continue in this fashion ad infinitum. What is the sum of the perimeters all the triangles?

    (a) 6
    (b) 3^2
    (c) 3^3
    (d) ∞
    (e) None of the above.

    Solution: Observe that the trangle formed by the midpoints of the larger trangle is similar with a ratio of $\frac{1}{2}$. Then the sequence $S_n = \frac{1}{2^n}$ describes the side length of the nth triangle. Then the perimeter of the nth triangle is $3(\frac{1}{2^n})$, because every triangle is equilateral. Now, notice that the sum of the perimeters of all the triangles is a geometric sum; hence, we have

    $$\sum_{n=0}^{\infty} 3S_n = \sum_{n=0}^{\infty} 3\frac{1}{2^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 3\left(\frac{1}{1-\frac{1}{2}}\right) = 3(2) = 6$$

    Answer. (a)

11. Determine the area enclosed by the quadrilateral ABCD where $|AB| = 3$, $|BC| = 9$, $|CD| = 7$, $|DA| = 5$, and $m\angle DAB = 120°$.

    (a) $\frac{2}{3}(5\sqrt{3} + 3\sqrt{115})$
    (b) $\frac{5}{7}(d\sqrt{3} + 3\sqrt{115})$
    (c) $\frac{3}{5}(5\sqrt{3} + 3\sqrt{115})$
    (d) $\frac{3}{4}(5\sqrt{3} + 3\sqrt{115})$
    (e) None of these

    Solution. First draw line segment BD. Notice that it divides quadrilateral

$ABCD$ into two triangles: ABD and BDC. We can find $|BD|$ using the Law of cosines:
$|BD|^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos(120°)$
$|BD|^2 = 34 - 30\cos(120°)$
$|BD|^2 = 34 - 30(\frac{-1}{2})$
$|BD|^2 = 49$
$|BD| = 7$
Now use Heron's formula to compute the areas of both triangles.
Area $(\triangle ABD) = \dfrac{15\sqrt{3}}{4}$
Area $(\triangle BDC) = \dfrac{9\sqrt{115}}{4}$
The total area is:
$\dfrac{15\sqrt{3}}{4} + \dfrac{9\sqrt{115}}{4} = \dfrac{3}{4}(5\sqrt{3} + 3\sqrt{115})$.
Answer. (d)

12. (MH 11-12, 2009) Two circles of radius 2 are drawn so that each circle passes through the center of the other. What is the perimeter of the region of overlap?
    (a) $\frac{4\pi}{3}$
    (b) 4π
    (c) 2π
    (d) $\frac{8\pi}{3}$
    (e) None of the above.

    Solution. Notice that the points of intersection of the circles and their centers form equilateral triangles. Thus, the arc measure between the points of intersection is 120°. Now, we will calculate the corresponding fraction of the circumference of one circle:

    $$\text{Half of desired perimeter} = \tfrac{120°}{360°}(2\pi r) = \tfrac{1}{3}(2 \cdot 2\pi) = \tfrac{4\pi}{3}$$

    Now, doubling this, we get that the perimeter of the region of overlap is $\frac{8\pi}{3}$.
    Answer. (d)

13. (MH 11-12, 2010) The edge of a cube is 2 in. What is the distance between the two opposite corners of the cube?
    (a) $2\sqrt{6}$ in
    (b) $2\sqrt{5}$ in
    (c) $2\sqrt{3}$ in
    (d) $2\sqrt{2}$ in
    (e) None of the above

    Solution. Let us form a right triangle by connecting opposite corners of a face (a diagonal) and the adjacent edge such that the length determined by the two opposite corners of the cube is the hypotenuse of the right angle formed by the first two segments. Now, let d denote the distance between the two opposite corners of the cube. Thus, by the Pythagorean Theorem, we have

$$d^2 = (2\sqrt{2})^2 + 2^2$$
$$d^2 = 8 + 4$$
$$d^2 = 12$$
$$d = \sqrt{12} = 2\sqrt{3}$$

Answer. (c)

14. Consider five squares s_1, s_2, s_3, s_4, and s_5, where s_1 is inscribed in s_2, s_2 is inscribed in s_3, s_3 is inscribed in s_4, and s_4 is inscribed in s_5. Each square touches its inscribed square so that the circumscribing squares sides are cut in ratios of 2:1. What is the ratio of the smallest square's area to the largest square's area?

    (a) $\dfrac{625}{6561}$

    (b) $\dfrac{16}{81}$

    (c) $\dfrac{1}{16}$

    (d) $\dfrac{1}{9}$

    (e) None of these

    Solution. Consider one pair of mutually circumscribing/inscribing squares. Let a be the length of the smaller square's side and b the length of the larger square's side. By looking at the right triangle whose hypotenuse is the smaller square's side and whose two legs are formed from a pair of adjacent sides of the larger square, we can use the Pythagorean Theorem to get the following equation:
    $\left(\dfrac{b}{3}\right)^2 + \left(\dfrac{2b}{3}\right)^2 = a^2$
    Solve for a^2:
    $a^2 = \dfrac{5}{9}b^2$
    $\dfrac{a^2}{b^2} = \dfrac{5}{9}$
    So the ratio of the area of the smaller square area to the larger square area is $\frac{5}{9}$. Thus the ratio of the smallest square's area to the largest square's area is
    $\left(\dfrac{5}{9}\right)^4 = \dfrac{625}{6561}$.
    Answer. (a)