Problem Solving Session (aka MFD prep)
CSU Fresno
March 7, 2015
Topics: Number theory

Problem Solving Sessions website:
`http://zimmer.csufresno.edu/~mnogin/mfd-prep.html`

Math Field Day date: Saturday, April 18, 2015

Math Field Day website:
`http://www.fresnostate.edu/csm/math/news-and-events/field-day/`

Mini Mad Hatter (individual, 2 minutes per problem)

1. (MH 2014 11-12) For what value of n is it true that

$$3^1 \cdot 3^2 \cdot 3^3 \cdots 3^n = 3^{253}?$$

   (a) $n = 20$
   (b) $n = 21$
   (c) $n = 22$
   (d) $n = 23$
   (e) No value of n

   Solution. (c)
   $3^{1+2+3+\ldots+n} = 3^{253}$
   $1 + 2 + 3 + \ldots + n = 253$
   $\frac{n(n+1)}{2} = 253$
   $n(n+1) = 506$
   Solving $n^2 + n - 506 = 0$ using the quadratic formula gives $n = 22$ or $n = -23$.
   Alternatively, note the last digit of the product is 6. From the above answer choices, only for $n = 22$ the last digit of $n(n+1)$ is 6. Indeed, $22 \cdot 23 = 506$.

2. (MH 2014 9-10) Determine the sum $7 + 11 + 15 + \ldots + 83$.

   (a) 903
   (b) 900
   (c) 820
   (d) 630

   Solution. (b)
   $7 + 11 + 15 + \ldots + 75 + 79 + 83 =?$
   Approach 1.
   Pair up the numbers: $7 + 83 = 90$, $11 + 79 = 90$, $15 + 75 = 90$.
   The distance between 7 and 83 is $76 = 19 \cdot 4$, so there are 20 numbers.

Thus there are 10 pairs.
$7 + 11 + 15 + \ldots + 75 + 79 + 83 = 10 \cdot 90 = 900$.
Approach 2.
Modify the given sum. First subtract 3 from each term:
$4 + 8 + 12 + \ldots + 72 + 76 + 80 =?$
Now divide each term by 4:
$1 + 2 + 3 + \ldots + 18 + 19 + 20 = \frac{20 \cdot 21}{2} = 210$
Then $4 + 8 + 12 + \ldots + 72 + 76 + 80 = 4 \cdot 210 = 840$,
and $7 + 11 + 15 + \ldots + 75 + 79 + 83 = 840 + 20 \cdot 3 = 900$.

3. (MH 2014 11-12) What is the number of integers n for which $\frac{7n+15}{n-3}$ is an integer?
   (a) 6
   (b) 9
   (c) 12
   (d) 18
   (e) 24

   Solution. (d)
   $\frac{7n+15}{n-3} = \frac{7(n-3)+36}{n-3} = 7 + \frac{36}{n-3}$.
   This value is an integer when $(n-3)|36$.
   The number 36 has 9 positive factors: 1, 2, 3, 4, 6, 9, 12, 18, 36.
   (Note: we can also use the prime factorization $36 = 2^2 3^2$ to calculate the number of positive factors).
   There are also 9 negative factors, so 18 factors total.
   Thus $(n-3)|36$ has 18 integer solutions.

4. (MH 2014 9-10) Find the first of three consecutive odd integers whose sum is 57.
   (a) 13
   (b) 15
   (c) 17
   (d) 19

   Solution. (c)
   If $2n + 1$, $2n + 3$, $2n + 5$ are three consecutve odd integers, their sum is $6n + 9$.
   Solving $6n + 9 = 57$ gives $n = 8$. So the first number is $2n + 1 = 17$.
   Alternatively, we can write the three consecutive odd integers as $a - 2$, a, and $a + 2$, then their sum is $(a - 2) + a + (a + 2) = 3a$, i.e. 3 times the middle number. Thus the middle number is $57/3 = 19$. The first number is then 17.
   Finally, we can just check the given answer choices:
   $13 + 15 + 17 \neq 57$, $15 + 17 + 19 \neq 57$, $17 + 19 + 21 = 57$.

5. (MH 2014 9-10) If the product of five integers is a multiple of 32, then what is the smallest number of these integers that must be even?
   (a) 5

(b) 4

(c) 2

(d) 1

Solution. (d)
Only one must be an even number. If one number is a multiple of 32 and the other 4 numbers are any integers, then the product is a multiple of 32.

6. (MH 2014 9-10) For how many integers n is $n^2 - 30$ negative?

   (a) 5

   (b) 6

   (c) 11

   (d) infinitely many

   Solution. (c)
   $n^2 - 30 < 0$
   $n^2 < 30$
   Integer solutions are: $-5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5.$

7. (MH 2010 11-12) Which of the following numbers is a perfect square?

   (a) 98!99!

   (b) 98!100!

   (c) 99!100!

   (d) 99!101!

   (e) 100!101!

   Solution. (c)
   $98!99! = (98!)^2 \cdot 99$ – not a perfect square
   $98!100! = (98!)^2 \cdot 99 \cdot 100$ – not a perfect square
   $99!100! = (99!)^2 \cdot 100 = (99!)^2 \cdot 10^2 = (99! \cdot 10)^2$
   Answers (d) and (e) are not perfect squares (verify this similarly to (a) and (b)).

8. (MH 2012 11-12) How many different prime factors does the number 20! have?

   (a) 6

   (b) 8

   (c) 19

   (d) 20

   (e) None of the above

   Solution. (b)
   The prime factors of 20! are all the prime numbers between 1 and 20:
   2, 3, 5, 7, 11, 13, 17, 19. So there are 8 of them.

Mini Leap Frog (2 participants per team)

1. (LF 2014 9-10) The 9-digit number $N = 1234d4321$ is divisible by 9. What is the value of the digit d?
   (a) 7
   (b) 6
   (c) 5
   (d) 4
   (e) None of these

   Solution. (a)
   A natural number is divisible by 9 if and only if the sum of its digits is divisible by 9.
   $1 + 2 + 3 + 4 + d + 4 + 3 + 2 + 1 = 20 + d$ is divisible by 9 when $d = 7$.

2. (LF 2014 11-12) Given that $2^{60} = 1,152,921,504,606,846,976$, find the first four digits (reading left to right) of 2^{61} and 2^{59}, then add these 8 digits up to get:
   (a) 35
   (b) 30
   (c) 32
   (d) 28
   (e) None of the above

   Solution. (c)
   $2^{61} = 2,305,843,\ldots$
   $2^{59} = 576,460,\ldots$
   $2 + 3 + 0 + 5 + 5 + 7 + 6 + 4 = 32$

3. (LF 2013 9-10) How many 4-digit palindromic numbers *abba* are divisible by 9?
   (a) 7
   (b) 8
   (c) 9
   (d) 10
   (e) None of these

   Solution. (d)
   $9|abba$ iff $9|2(a + b)$ iff $9|(a + b)$.
   Since $a \neq 0$, $a + b = 9$ or $a + b = 18$.
   $a + b = 9$ has 9 solutions: $(1, 8)$, $(2, 7)$, …, $(9, 0)$.
   $a + b = 18$ has 1 solution: $(9, 9)$.

4. (LF 2014 9-10) The sum of the prime divisors of 2014 is ….
   (a) 76

(b) 78

(c) 80

(d) 82

(e) None of these

Solution. (e)
$2014 = 2 \cdot 1007 = 2 \cdot 19 \cdot 53$.
$2 + 19 + 53 = 74$.

5. (LF 2013 11-12) How many integers of the form $n^4 + 4$, where n is a non-negative integer, are prime? *Hint:* Complete the square.

   (a) 0

   (b) 1

   (c) 2

   (d) 3

   (e) 4

   Solution. (b)
   $n^4+4 = (n^2+2)^2-4n^2 = (n^2+2)^2-(2n)^2 = (n^2+2+2n)(n^2+2-2n) = ((n+1)^2+1)((n-1)^2+1)$.
   For any $n \geq 2$ each factor is greater than 1, so the product is not prime.
   If $n = 0$, $n^4 + 4 = 4$ is not prime.
   If $n = 1$, $n^4 + 4 = 5$ is prime.

6. (LF 2014 9-10) The digit sum of a number is the sum of its decimal digits. For example, the digit sum of the number 3206 is $3 + 2 + 0 + 6 = 11$. Determine the digit sum of the number $(10^{2014} + 1)^4$.

   (a) 10

   (b) 12

   (c) 14

   (d) 16

   (e) None of these

   Solution. (d)
   $(10^{2014} + 1)^4 = (10^{2014})^4 + \binom{4}{1}(10^{2014})^3 + \binom{4}{2}(10^{2014})^2 + \binom{4}{3}10^{2014} + 1 =$
   $10^{8056} + 4 \cdot 10^{6042} + 6 \cdot 10^{4028} + 4 \cdot 10^{2014} + 1$
   The non-zero digits of this number are 1, 4, 6, 4, 1. Their sum is 16.
   Alternatively, $(10^{2014} + 1)^4 \equiv (1^{2014} + 1)^4 \equiv 2^4 \equiv 16 \pmod 9$. So the answers (a), (b), (c) can be eliminated. Now we have only (d) and (e) left.

More practice problems

1. (MH 2012 11-12) If a, b, $a + b$, and $a - b$ are all prime numbers, which of the following

statements must be true about the sum of these four numbers?

(a) The sum is odd and prime.

(b) The sum is odd and divisible by 3.

(c) The sum is odd and divisible by 7.

(d) The sum is even.

(e) None of the above.

Solution. (a)
First notice that a and b cannot be both odd, otherwise $a+b$ is even larger than 2, so not prime. So either $a=2$ or $b=2$.
Since $a-b$ must be prime, $a>b+1$. So $b=2$ and a is odd, $a \geq 5$.
Next, of the three numbers $a-2$, a, and $a+2$ one must be divisible by 3. Since it must be prime, it must be equal to 3.
So $a-b=a-2=3$, thus $a=5$ and $a+b=7$.
Then the sum is then $a+b+(a+b)+(a-b)=5+2+7+3=17$. It is odd and prime.

2. (LF 2014 11-12) Let r be the remainder of

$$1+2^2+3^3+4^4+5^5+6^6+7^7+8^8+9^9+10^{10}$$

when divided by 3. Let s be the sum of the last digits of each of the terms of the sum above. What is $r+s$?

(a) 47

(b) 49

(c) 45

(d) 42

(e) None of the above

Solution. (b)
$1+2^2+3^3+4^4+5^5+6^6+7^7+8^8+9^9+10^{10} \equiv 1+1+0+1+2+0+1+1+0+1 \equiv 8 \equiv 2 \pmod 3$,
$1+4+7+6+5+6+3+6+9+0=47$,
$2+47=49$

3. (MH 2014 11-12) Out of all relatively prime integers a and b, what is the largest possible value of the greatest common divisor of $a+201b$ and $201a+b$?

(a) 37542

(b) 39264

(c) 40400

(d) 42176

(e) 44862

Solution. (c)
Let d be the greates common divisor of $a+201b$ and $201a+b$.

Both $201(201a+b)-(a+201b) = 201^2a - a = (201^2-1)a = 40400a$ and
$201(a+201b)-(201a+b) = 201^2b - b = (201^2-1)b = 40400b$ are divisible by d.
Since a and b are relatively prime, 40400 must be divisible by d.

4. (MH 2014 11-12) Given that $29a031 \times 342 = 100900b02$ where a, b denote missing digits, what is the value of $a+b$?

   (a) 7

   (b) 8

   (c) 9

   (d) 10

   (e) 11

   Solution. (e)
   Since 9 is a factor of 342, it must be a factor of $100900b02$. Then $1+9+b+2 = 12+b$ must be divisible by 9. So $b=6$.
   Now, $\frac{100900602}{342} = 295031$, so $a=5$.
   $a+b = 5+6 = 11$.

5. (MH 2014 11-12) What is the sum of the digits of $(1010101)^2$?

   (a) 12

   (b) 16

   (c) 17

   (d) 19

   (e) 21

   Solution. (b)
   $(1010101)^2 = 1020304030201$, the sum of the digits is 16.
   Alternatively, the sum of the digits is congruent to the number itself modulo 9, and $(1010101)^2 \equiv 4^2 \equiv 16 \pmod 9$.
   Of the given answer choices only (b) satisfies this condition.

6. (MH 2014 11-12) How many positive integers less than 1200 have no repeating digits, i.e. no digit occurs more than once?

   (a) 648

   (b) 682

   (c) 794

   (d) 812

   (e) 817

   Solution. (c)
   9 one-digit numbers: 1, 2, . . ., 9.
   81 two-digit numbers: the first digit has 9 choices (1-9), and the second digit has 9 choices (any digit other than the first one).

$9 \cdot 9 \cdot 8 = 648$ three-digit numbers: the first digit has 9 choices (1-9), the second digit has 9 choices (any digit other than the first one), and the third has 8 choices (any digit different from the first two).
$8 \cdot 7 = 56$ four-digit numbers: the first digit must be 1, the second ditig must be 0, the third digit has 8 choices, and the third digit has 7 choices.
So there are $9 + 81 + 648 + 56 = 794$ such numbers total.

7. (LF 2014 11-12) The smallest prime number that divides $2^{111} + 3^{111}$ is
   (a) 23
   (b) 2^{111}
   (c) 17
   (d) 3^{111}
   (e) None of the above

   Solution. (e)
   The last digits of 2^n repeat in the cycle: 2, 4, 8, 6.
   The last digits of 3^n repeat in the cycle: 3, 9, 7, 1.
   So the last digits of $2^n + 3^n$ repeat in the cycle: 5, 3, 5, 7.
   Notice that the last digit of $2^n + 3^n$ is 5 whenever n is odd.
   Alternatively, $a^n + b^n$ is divisible by $a + b$ for every odd n.
   So $2^{111} + 3^{111}$ is divisible by $2 + 3 = 5$.
   It is easy to see that $2^{111} + 3^{111}$ is not divisible by 2 or 3.
   So the smallest prime factor of $2^{111} + 3^{111}$ is 5.

8. (LF 2014 9-10) A standard calendar year has 365 days. A leap year has 366 days. A year is a leap year if it is divisible by 4, *except* if it is a new century year not divisible by 400. So 1900 was not a leap year (1900 is not divible by 400), but 2000 was a leap year (2000 is divisible by 400). December 25, 2013 was a Wednesday. What day of the week will December 25 be in the year 3013?
   (a) Friday
   (b) Saturday
   (c) Sunday
   (d) Monday
   (e) None of the above

   Solution. (b)
   If we divide 365 by 7, we get a remainder of 1. This means that without leap years, the days of the week for a particular calendar date would advance 1 day per year. There is 1000 years from 2013 to 3013, so this would account for 1000 added days. However, we must then add the 250 leap days (1000/4) less the non-leap days in the new century years not divisible by 400 (there are 8 of these), giving a total of $1000 + 250 - 8 = 1242$. Divide 1242 by 7 to get a remainder of 3. This means December 25, 3013 will occur 3 days after Wednesday, which is Saturday.

9. (MH 2014 11-12) Find the last two digits of 2014^{2014}.
   (a) 36
   (b) 16
   (c) 56
   (d) 76
   (e) 96

   Solution. (b)
   Finding the last two digits is equivalent to finding the remainder upon division by 100. So one strategy is to compute 2014^n modulo 100 for some small n and hope to see a pattern soon. This approach may require more than allowed two minutes though. So let's try to make use of some smaller moduli.
   Observe that $2014^{2014} \equiv 1 \pmod 3$, and the only answer choices that have a remainder of 1 upon division by 3 are (b) and (d).
   Modulo 5 would not distinguish 16 and 76, so it's useless here.
   Let's consider some other small modulo that will distinguish 16 and 76, for example, 7:
   $2014^{2014} \equiv 5^{2014} \equiv (5^3)^{671} \cdot 5 \equiv (-1)^{671} \cdot 5 \equiv -5 \equiv 2 \pmod 7$.
   Only 16 has a remander of 2 upon division by 7.