CSU Fresno Problem Solving Session

Problem Solving Sessions website: `http://zimmer.csufresno.edu/~mnogin/mfd-prep.html`

Math Field Day date: Saturday, April 17, 2010

Math Field Day website: `http://www.csufresno.edu/math/news_and_events/field_day/`

Number theory, 20 Feb 2010

Useful formulas and theorems, and hints

$1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$, proof by Gauss

Partial sum of geometric series: $1 + q + q^2 + \ldots + q^n = \frac{1-q^{n+1}}{1-q}$

Sum of infinite geometric series: $1 + q + q^2 + \ldots = \frac{1}{1-q}$

Difference of squares/cubes, sum of cubes, square/cube of sum/difference, etc.

Prime Factorization theorem: every positive integer can be written as a product of primes, uniquely up to order.

The number of positive divisors of $N = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$ is $(k_1 + 1)(k_2 + 1) \ldots (k_n + 1)$.

Division with remainder

May need to introduce notations, e.g. for n, $n+1$, $n+2$, etc. for consecutive integers, $2n/2n+1$ for even/odd integers, etc.

May need to perform some calculations, up to multiplying/dividing 2- or 3-digit numbers, or adding/subtracting 4-digit numbers, but usually not larger ones

Examples

1. (MH 11-12, 2006) Compute: $1 + 2 + 3 + \ldots + 2006$.
   (a) $4,017$
   (b) $2,012,018$
   (c) $2,013,021$
   (d) $4,024,036$
   (e) None of the above

   Solution. $1 + 2 + 3 + \ldots + 2006 = \frac{2006 \cdot 2007}{2} = 1003 \cdot 2007 = 20013021$.

2. If $a = 6$ and $b = 24$, find $\dfrac{a^8 - b^4}{(a^4 + b^2)(a^2 + b)}$.
   (a) 0

(b) 1

(c) 3

(d) 6

(e) 12

Solution. $\dfrac{a^8-b^4}{(a^4+b^2)(a^2+b)} = \dfrac{(a^4-b^2)(a^4+b^2)}{(a^4+b^2)(a^2+b)} = \dfrac{(a^2-b)(a^2+b)(a^4+b^2)}{(a^4+b^2)(a^2+b)} =$
$a^2 - b = 36 - 24 = 12$

3. (MH 11-12, 2006) How many different factors does the number 10! have?

   (a) 2^6

   (b) 2^{15}

   (c) 10

   (d) 270

   (e) None of the above

   Solution. $10! = 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$
   The number of factors is $9 \cdot 5 \cdot 3 \cdot 2 = 270$.

4. (MH 11-12, 2006) Today is Saturday. What day of the week will be exactly 2006 days from today?

   (a) Monday

   (b) Tuesday

   (c) Wednesday

   (d) Thursday

   (e) Friday

   Solution. $2006 = 286 * 7 + 4$. Answer: Wednesday.

Mad Hatter

1. (MH 9-10, 2006) Determine the sum $3 + 7 + 11 + \ldots + 35$.

   (a) 140

   (b) 171

   (c) 315

   (d) 342

   Solution. Let
   $$\begin{aligned} S &= \ \ 3 + \ \ 7 + 11 + \ldots + 35 \\ S &= 35 + 31 + 27 + \ldots + \ \ 3 \\ 2S &= 38 + 38 + 38 + \ldots + 38 = 38 \cdot 9 = 342. \end{aligned}$$

2. (MH 9-10, 2006) Suppose a positive integer N is divisible by both 9 and 21. What is the smallest possible number of positive integers that divide N?

(a) 6

(b) 5

(c) 4

(d) 3

Solution. N is divisible by both 3^2 and 7, so its prime factorization must contain at least $3^2 \cdot 7$. N will have the smallest possible number of positive divisors if $N = 3^2 \cdot 7$. It has $3 \cdot 2 = 6$ positive divisors.

3. (MH 11-12, 2006) Which of the numbers 2^{100}, 3^{50}, $10,000^2$, 500^3 is the largest?

   (a) 2^{100}

   (b) 3^{50}

   (c) $10,000^2$

   (d) 500^3

   Solution. $2^{100} = (2^2)^{50} = 4^{50} > 3^{50}$. So the answer is not (b).
   $2^{100} = (2^{10})^{10} = (1024)^{10} > (10^3)^{10} = 10^{30}$,
   $10,000^2 = 100,000,000$,
   $500^3 = 125,000,000$.
   Answer: (a).

4. (MH 11-12, 2006) Simplify: $\dfrac{\sqrt{4}+\sqrt{6}+\sqrt{24}}{\sqrt{1}+\sqrt{6}+\sqrt{9}+\sqrt{150}}$.

   (a) $\dfrac{1}{5}$

   (b) $\dfrac{1}{2}$

   (c) 1

   (d) $\dfrac{4}{15}$

   (e) None of the above

   Solution. $\dfrac{\sqrt{4}+\sqrt{6}+\sqrt{24}}{\sqrt{1}+\sqrt{6}+\sqrt{9}+\sqrt{150}} = \dfrac{2+\sqrt{6}+2\sqrt{6}}{1+\sqrt{6}+3+5\sqrt{6}} = \dfrac{2+3\sqrt{6}}{4+6\sqrt{6}} = \dfrac{1}{2}$.

5. (MH 11-12, MH 9-10, 2006) Evaluate: $\dfrac{4351^2-4347^2}{4350\cdot 4353-4351^2}$.

   (a) $\dfrac{1}{2}$

   (b) 1

   (c) 2

   (d) 4

   (e) 8

Solution. Let $4350 = n$. Then $\dfrac{4351^2 - 4347^2}{4350 \cdot 4353 - 4351^2} = \dfrac{(n+1)^2 - (n-3)^2}{n(n+3) - (n+1)^2} = \dfrac{n^2 + 2n + 1 - n^2 + 6n - 9}{n^2 + 3n - n^2 - 2n - 1} = \dfrac{8n - 8}{n - 1} = 8.$

6. (MH 11-12, 2006) How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?
   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) 5

   Solution. An integer has 15, 20, and 25 as factors when 3, 5^2, and 2^2 its factors, i.e. the integer is divisible by $3 \cdot 5^2 \cdot 2^2 = 3 \cdot 25 \cdot 4 = 300$. Integers between 1000 and 2000 that are divisible by 300 are: 1200, 1500, and 1800. Answer: 3

7. (MH 9-10, 2006) If a, b, $a + b$, and $a - b$ are all prime numbers, which of the following statements must be true about the sum of these four numbers?
   (a) The sum is odd and prime.
   (b) The sum is odd and divisible by 3.
   (c) The sum is odd and divisible by 7.
   (d) The sum is even.

   Solution. $a + b + a + b + a - b = 3a + b$. None of the above is obviously true or false from this expression, so let's try to find some information about a and b.
   If both a and b are odd, then $a + b$ is even and at least 6, so not prime.
   If $a = b = 2$, $a + b = 4$ and is not prime.
   So a is an odd prime, and $b = 2$. So $3a+b = 3a+2$ where a is odd. This eliminates answer choices (b) and (d).
   Further, $a \neq 3$ since $3 - 2 = 1$ is not prime. But $a = 5$ gives $a - b = 3$ and $a + b = 7$, so all four numbers are prime. Thus $3a + b = 17$. Answer: (a)
   Alternatively, one of a, $a+2$, $a-2$ is divisible by 3. Since it is also prime, it must be equal to 3. So $a - 2 = 3$, and $a = 5$. Proceed as above.

8. (MH 9-10, 2006) How many 2-digit numbers, $n \geq 10$ are there such that both digits are squares (e.g., 10 and 41 are two such numbers)?
   (a) 3
   (b) 6
   (c) 8
   (d) 12

   Solution. The first digit can be 1, 4, or 9. The second digit can be 0, 1, 4, or 9. There are $3 \cdot 4 = 12$ such numbers.

Leap Frog

1. (LF 9-12, 2006) The units digit of the number $9^{2006} - 3^{2006}$ is

   (a) 6

   (b) 4

   (c) 2

   (d) 0

   (e) None of these

   Solution. Since $9^2 = 81$ and the units digit of any power of 81 is 1,
   $9^{2006} = (9^2)^{1003}$ has units digit 1, and
   $3^{2006} = (3^2)^{1003} = 9^{1003} = 9 \cdot 9^{1002} = 9 \cdot (9^2)^{501}$ has units digit 9.
   The units digit of the difference is 2.

2. (LF 9-12, 2006) If the natural numbers 0, 1, 2, ... are lined up to form a non-ending string of digits

   $$01234567891011121314\ldots$$

   then the 2006th digit in the string is

   (a) 9

   (b) 0

   (c) 3

   (d) 7

   (e) None of these

   Solution. The numbers 0 through 9 give 10 digits.
   The numbers 10 through 99 give 180 digits.
   So the numbers 0 through 99 contribute 190 digits.
   We need 1816 more digits.
   Since $1816 = 605 \times 3 + 1$, we know we can accomplish this with the 1st digit of the 606th number after 99. This number is 705. So the 2006th digit in the sequence is 7.

3. (LF 9-12, 2006) Suppose N is the 3-digit number $N = abc$ where a, b and c are the respective hundreds, tens and units digits. Assuming that when N^2 is divided by 100 the remainder is 1, then what are the possible values for b?

   (a) $b = 0$, 4, 5, or 9

   (b) $b = 0$, 4, 5, 6, or 9

   (c) $b = 0$, 4, 5, or 8

   (d) $b = 0$, 4, 6, 8, or 9

   (e) None of these

   Solution.

$$
\begin{array}{ccccc}
 & & a & b & c \\
 & \times & a & b & c \\
\hline
 & & ac & bc & c^2 \\
 & ab & b^2 & bc & \\
a^2 & ab & ac & & \\
\hline
 & \dots & & 2bc & c^2
\end{array}
$$

We first observe that the answer is independent of a. This is because $N = 100a + (bc)$ and so $N^2 = 10000a^2 + 200a(bc) + (bc)^2$. Since $10000a^2 + 200a(bc)$ is divisible by 100, the remainder of N is the same as the remainder of (bc). Thus, in our analysis we may replace N by $N' = bc = 10b + c$. Square N' to get

$$
\begin{aligned}
N'^2 &= (10b + c)^2 \\
&= 100b^2 + 20bc + c^2.
\end{aligned}
$$

The $100b^2$ term will not contribute to the remainder, and so may be ignored. We are then left with dividing $20bc + c^2$ by 100 to get a remainder of 1. Clearly this means $c = 1$ or 9.
If $c = 1$, then $20bc + c^2 = 20b + 1$, which by checking the possibilities exhaustively, will have a remainder of 1 only if $b = 0$ or 5. And this can happen if say $N = 101$ or 151.
Similarly, if $c = 9$, then $20bc + c^2 = 180b + 81$, which will have a remainder of 1 only if $b = 4$ or 9. This can happen if say $N = 149$ or $N = 199$.
So the possible values for b are 0, 4, 5 or 9.

4. (LF 9-12, 2006) Suppose n, a and b are positive integers. In order for n to divide ab, it is _______________ that n divides a or n divides b.

   (a) necessary and sufficient

   (b) necessary, but not sufficient

   (c) sufficient, but not necessary

   (d) neither necessary nor sufficient

   (e) None of these

   Solution. It is not necessary because 4 divides $4 = 2 \times 2$, but 4 does not divide 2. It is sufficient however. This is because if say n divides say a, then $a = nm$ for some other positive integer m and hence $ab = nmb$ which is clearly divisible by n. Answer: (c)

5. (LF 9-12, 2006) Let $(0.xyxyxy\ldots)_b$ and $(0.yxyxyx\ldots)_b$ be the base b representations of the two numbers A and B respecitvely, where x and y represent base b digits, not both of which are zero. Then,

$$\frac{A}{B} =$$

   (a) $\dfrac{y+b}{x+b}$

(b) $\dfrac{x+b}{y+b}$

(c) $\dfrac{xb+y}{yb+x}$

(d) $\dfrac{yb+x}{xb+y}$

(e) None of these

Solution 1. Let's first convert A to base 10.

$$\begin{aligned}
A &= \frac{x}{b}+\frac{y}{b^2}+\frac{x}{b^3}+\frac{y}{b^4}+\cdots \\
&= \left(\frac{x}{b}+\frac{x}{b^3}+\cdots\right)+\left(\frac{y}{b^2}+\frac{y}{b^4}+\cdots\right) \\
&= \frac{x}{b}\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\cdots\right)+\frac{y}{b^2}\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\cdots\right) \\
&= \left(\frac{x}{b}+\frac{y}{b^2}\right)\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\cdots\right) \\
&= \frac{bx+y}{b^2}\cdot\frac{1}{1-\frac{1}{b^2}} \\
&= \frac{bx+y}{b^2-1}
\end{aligned}$$

By reversing the roles of x and y, we get $B=\dfrac{by+x}{b^2-1}$ and so

$$\frac{A}{B}=\frac{bx+y}{by+x}.$$

Solution 2. From the given base-b representations we get two equations (1) $bA=B+x$ and (2) $bB=A+y$. Equation (1) gives $B=bA-x$, and substituting this into Equation (2) gives

$$\begin{aligned}
b(bA-x)=A+y &\implies b^2A-bx=A+y \\
&\implies (b^2-1)A=bx+y \\
&\implies A=\frac{bx+y}{b^2-1}.
\end{aligned}$$

Then proceed as in solution 1.

6. (MH 9-10, 2006) For how many positive integers m does there exist at least one integer n such that $mn\le m+n$?

(a) 4

(b) 6

(c) 12

(d) infinitely many

Solution. If $n=0$, $0\le m$ for any positive integer m. Answer: infinitely many.

More Problems

1. (MH 11-12, 2006) Simplify: $\dfrac{123 \cdot 456 + 123 + 456}{123 + 456 \cdot 124} =$

   (a) 0.5

   (b) 1

   (c) 2

   (d) $\dfrac{123}{124}$

   (e) None of the above

   Solution. $\dfrac{123 \cdot 456 + 123 + 456}{123 + 456 \cdot 124} = \dfrac{(123 \cdot 456 + 456) + 123}{123 + 456 \cdot 124} =$
   $\dfrac{(123 + 1) \cdot 456 + 123}{123 + 456 \cdot 124} = \dfrac{124 \cdot 456 + 123}{123 + 456 \cdot 124} = 1$

2. (MH 9-10, 2006) Which of the following is the largest number?

   (a) $2^{(3^4)}$

   (b) $4^{(3^2)}$

   (c) $8^{(4^2)}$

   (d) $(16^8)^2$

   Solution. Since all bases are powers of 2, express each number as a power of 2:
   $2^{(3^4)} = 2^{81}$,
   $4^{(3^2)} = 4^9 = (2^2)^9 = 2^{18}$,
   $8^{(4^2)} = 8^{16} = (2^3)^{16} = 2^{48}$,
   $(16^8)^2 = 16^{16} = (2^4)^{16} = 2^{64}$.
   Answer: (a)

3. (MH 11-12, 2006) Find the smallest positive prime p such that p^3+10p^2 is a perfect square. What is the sum of the digits of p?

   (a) 7

   (b) 8

   (c) 12

   (d) 14

   (e) None of the above

   Solution. $p^3 + 10p^2 = p^2(p + 10)$, so $p + 10$ must be a perfect square. Checking the values $p + 10 = 16, 25, 36, 49, 64, 81$ we see that the smallest prime p that has this property is 71. The sum of its digits is 8.

4. (MH 11-12, 2006) Approximately what percentage of the first $10,000$ natural numbers have a 1 somewhere in them?

   (a) 10%

   (b) 22%

(c) 34%

(d) 45%

(e) None of the above

Solution. Count the number of integers from 1 to $9,999$ that do not have a 1, i.e. consist entirely of digits 0, 2, 3, *ldots*,, 9, i.e. each of the four digits has 9 choices. There are 9^4 such integers, and $9^4 = (9^2)^2 = 81^2 = 6561$, or approximately 66%. Therefore approximately 34% have a 1 somewhere in them.

5. (MH 11-12, 2006) The sum of seven consecutive numbers is 126. Find the product of the smallest and the largest of these numbers.

   (a) 315

   (b) 324

   (c) 450

   (d) 468

   (e) None of the above

   Solution. $(n-3)+(n-2)+(n-1)+n+(n+1)+(n+2)+(n+3) = 126$
   $7n = 126$
   $n = 18$
   $n-3 = 15$, $n+3 = 21$, $15 \cdot 21 = 315$

6. (MH 11-12, 2006) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots+\dfrac{1}{2005\cdot 2006}=$

   (a) $\dfrac{2005}{2}$

   (b) 1003

   (c) $\dfrac{2005}{2006}$

   (d) $1-\dfrac{1}{2005\cdot 2006}$

   (e) None of the above

   Solution. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots+\dfrac{1}{2005\cdot 2006}=$
   $\left(1-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\ldots+\left(\dfrac{1}{2005}-\dfrac{1}{2006}\right)=1-\dfrac{1}{2006}=\dfrac{2005}{2006}$

7. (MH 9-10, 2006) What is $\sqrt{30}$ to the nearest hundredth?

   (a) 5.25

   (b) 5.38

   (c) 5.48

   (d) 5.77

   Solution. $5.4^2 = 29.16$, $5.5^2 = 30.25$, so the answer must be 5.48.

8. (MH 9-10, 2006) What is the greatest common divisor of 1776 and 1976?
   (a) 16
   (b) 8
   (c) 4
   (d) 2

   Solution. Since all answer choices are powers of 2, we only have to check factors that are powers of 2.
   $1776 = 2 \cdot 888 = 2 \cdot 8 \cdot 111 = 16 \cdot 111$.
   $1976 = 2 \cdot 988 = 4 \cdot 494 = 8 \cdot 247$. Answer: 8

9. (MH 9-10, 2006) What is the least common multiple of 36 and 243?
   (a) 8748
   (b) 2916
   (c) 972
   (d) 108

   Solution. $36 = 2^2 \cdot 3^2, \;\; 243 = 3^5$.
   $\text{LCM} = 2^2 \cdot 3^5 = 4 \cdot 243 = 972$.

10. (MH 9-10, 2006) What is the smallest positive prime p greater than 2 such that $p^3 + 7p^2$ is a perfect square?
    (a) 13
    (b) 17
    (c) 23
    (d) 29

    Solution. $p^3 + 7p^2 = p^2(p+7)$, so $p+7$ must be a perfect squre. Checking values $p + 7 = 9, 16, 25, 36$ we see that the smallest such prime p greater than 2 is 29.

11. (MH 9-10, 2006) If three distinct counting numbers have a sum of 10 and a product of 20, what is the median of the three numbers?
    (a) 3
    (b) 4
    (c) 5
    (d) There is not enough information given.

    Solution. $20 = 2 \cdot 2 \cdot 5$, and all numbers are less than 10 and distinct, the only possibility is 1, 4, and 5. Their sum is indeed 10 (just checking!), and the median is 4.

12. (LF 9-12, 2006) The sum of the prime divisors (each distinct prime divisor counted once regardless of exponent) of 1776 is
    (a) 19

(b) 21

(c) 113

(d) 42

(e) None of these

Solution. $1776 = 2^4 \times 3 \times 37$. So the sum of the prime divisors is $2+3+37 = 42$.

13. (LF 9-12, 2006) Suppose that the average of 10 numbers is 90. One of the numbers in the list is deleted, and the resulting average of nine numbers is equal to 91. What is the value of the deleted number?

(a) 80

(b) 81

(c) 82

(d) 83

(e) None of these

Solution. Let the 10 numbers be denoted by $N_1, N_2, \ldots, N_{10}$, where N_{10} will be the deleted number. We are given that the average of the original 10 numbers is 90. Thus

$$\frac{N_1 + N_2 + \cdots + N_{10}}{10} = 90,$$

which impies $N_1 + N_2 + \cdots + N_{10} = 900$. We are also given that

$$\frac{N_1 + N_2 + \cdots + N_9}{9} = 91,$$

and so $N_1 + N_2 + \cdots + N_9 = 819$. Putting this together gives

$$819 + N_{10} = 900$$

and so $N_{10} = 81$.