Problem Solving Session (aka MFD prep)
CSU Fresno
March 14, 2015
Topics: The number π, Area and Volume

Problem Solving Sessions website:
`http://zimmer.csufresno.edu/~mnogin/mfd-prep.html`

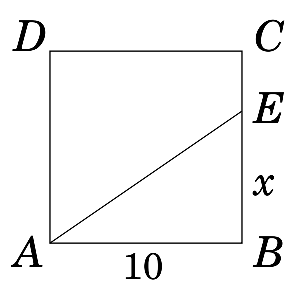
Math Field Day date: Saturday, April 18, 2015

Math Field Day website:
`http://www.fresnostate.edu/csm/math/news-and-events/field-day/`

Mini Mad Hatter (individual, 2 minutes per problem)

1. (MH 2014 9-10) Square $ABCD$ has side length 10. If point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40, what is BE?
   (a) 4
   (b) 5
   (c) 6
   (d) 8

   Solution.

   Area of triangle $= \frac{1}{2}bh$
   $40 = \frac{1}{2} \cdot 10 \cdot x$
   $40 = 5x$
   $x = 8$
   (d)

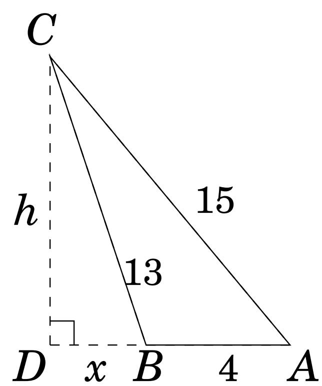
2. (MH 2014 9-10) If a and b are the length of the legs of a right triangle whose hypotenuse is 10 and whose area is 20, find $(a+b)^2$.
   (a) 180
   (b) 140
   (c) 120
   (d) 100

   Solution.

$a^2 + b^2 = 10^2 = 100$
$\frac{1}{2}ab = 20$
$ab = 40$
$(a+b)^2 = a^2 + 2ab + b^2 = 100 + 2 \cdot 40 = 180$
(a)

3. (MH 2014 11-12) Triangle ABC has $AC = 15$, $BC = 13$, and $AB = 4$. What is the length of the altitude from C to the extension of AB?
   (a) 9
   (b) 10
   (c) 11
   (d) 12
   (e) 13

   Solution 1.



Write the Pythagorean theorem for both right triangles:
$x^2 + h^2 = 13^2$
$(x+4)^2 + h^2 = 15^2$
Subtract the first equation from the second:
$(x+4)^2 - x^2 = 15^2 - 13^2$
$x^2 + 8x + 4^2 - x^2 = 225 - 169$
$8x + 16 = 56$
$8x = 40$
$x = 5$
Now the first equation becomes $25 + h^2 = 169$
$h^2 = 144$
$h = 12$
(d)
Solution 2.
Use Heron's formula ($A = \sqrt{\frac{p}{2}(\frac{p}{2} - a)(\frac{p}{2} - b)(\frac{p}{2} - c)}$) to find the area of triangle ABC:
$p = 15 + 13 + 4 = 32$, so $\frac{p}{2} = 16$,
$A = \sqrt{16 \cdot 3 \cdot 1 \cdot 12} = \sqrt{16}\sqrt{36} = 4 \cdot 6 = 24$.
On the other hand, $A = \frac{1}{2}bh = \frac{1}{2} \cdot 4h = 2h$, so
$2h = 24$
$h = 12$

4. (LF 2011 9-12) Lenny melts 2011 1″ by 1″ by 1″ ice cubes and refrezes the water to form one large ice cube (all side lengths equal). The side length of the large cube is
   (a) between 10 and 11 inches.
   (b) between 11 and 11 inches.
   (c) between 12 and 13 inches.
   (d) between 13 and 14 inches.
   (e) None of these

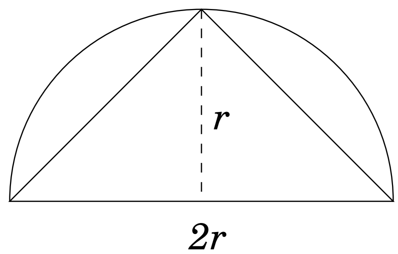
   Solution.
   The volume of the water is 2011 cubic inches. If x is the side length of the new cube, then $x^2 = 2011$.
   Since $12^3 = 1728$ and $13^3 = 2197$, x is between 12 and 13 inches.
   (c)

5. (MH 2014 9-10) A gold bar is a rectangular solid measuring $2 \times 3 \times 4$. It is melted down, and three cubes of equal size are constructed from this mold. What is the length of a side of each cube?
   (a) 8
   (b) 6
   (c) 4
   (d) 2

   Solution.
   The volume of the rectangular bar is $2 \cdot 3 \cdot 4 = 24$.
   The volume of each of the three cubes is $\frac{1}{3} \cdot 24 = 8$.
   The length of a side of each cube is 2 (since $2^3 = 8$).
   (d)

6. (MH 2014 9-10) The largest area of a triangle that can be inscribed in a semicircle of radius r is
   (a) $2r^2$
   (b) r^2
   (c) $\frac{1}{2}r^2$
   (d) $\frac{1}{4}r^2$

   Solution.
   The area of the semicircle is $\frac{1}{2}\pi r^2 < 2r^2$, so the area of an inscribed triangle cannot be $2r^2$.
   The area of r^2 is possible:

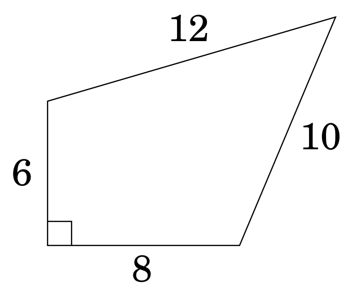


(b)

7. (MH 2014 9-10) A right circular cylinder has a radius of 8 and height of π^2. If a cube has the same volume as the cylinder, what is the length of an edge of the cube?
   (a) $4\sqrt{\pi}$
   (b) $8\sqrt{\pi}$
   (c) $4\pi\sqrt{\pi}$
   (d) 4π

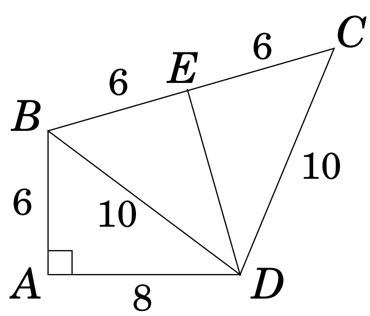
   Solution.
   The volume of the cylinder if $\pi r^2 h = \pi \cdot 64\pi^2 = 64\pi^3$.
   If x is the length of an edge of the cube, then $x^3 = 64\pi^3$, so $x = 4\pi$
   (d)

8. (MH 2012 11-12) Find the area of the quadrilateral shown below.



   (a) 48
   (b) 64
   (c) 72
   (d) 80
   (e) None of the above

   Solution.

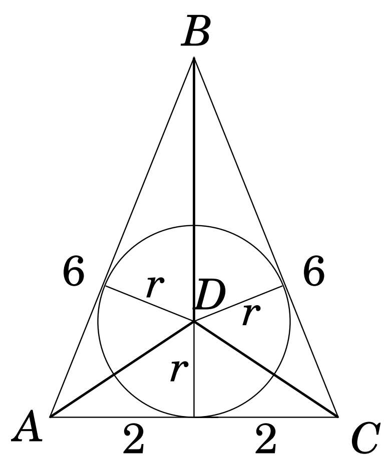


Draw the diagonal BD. The Pythagorean theorem for triangle ABD gives $BD = 10$.
Nocice that triangle BCD is isosceles. Then its height DE is also its median.
$BE = EC = 6$, so $DE = 8$.
Thus the quadrilateral consists of three congruent triangles.
The area of each triangle is $\frac{1}{2} \cdot 8 \cdot 6 = 24$.
The area of the quadrilateral is $3 \cdot 24 = 72$.
(c)

Mini Leap Frog (2 participants per team)

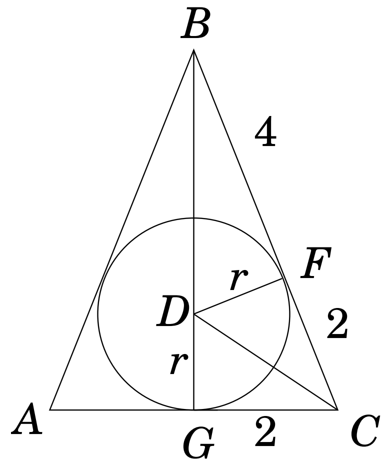
1. (LF 2013 9-10) A circle is inscribed in the isosceles triangle with side lengths 6, 6 and 4. Determine the area of the inscribed circle.
   (a) $\frac{\pi}{2}$
   (b) $\frac{3\pi}{2}$
   (c) $\frac{5\pi}{2}$
   (d) $\frac{7\pi}{2}$
   (e) None of these

Solution 1.



The area of trianlge ABC is the sum of the areas of triangles ABD, BCD, and ACD. These triangles have bases 6, 6, and 4, respectively, and all have height r.
So the area of triangle ABC is $\frac{1}{2} \cdot 6 \cdot r + \frac{1}{2} \cdot 6 \cdot r + \frac{1}{2} \cdot 4 \cdot r = 8r$.
On the other hand, the area of triangle ABC is $\frac{1}{2} \cdot b \cdot h$ where b and h are its base and height, respectively.
The height h can be found using the Pythagorean theorem:
$h^2 + 2^2 = 6^2$
$h^2 = 32$
$h = 4\sqrt{2}$
So the area of triangle ABC is $\frac{1}{2} \cdot 4 \cdot 4\sqrt{2} = 8\sqrt{2}$.
We have $8r = 8\sqrt{2}$, so $r = \sqrt{2}$.
The area of the circle is $\pi r^2 = 2\pi$.
(e)

Solution 2.

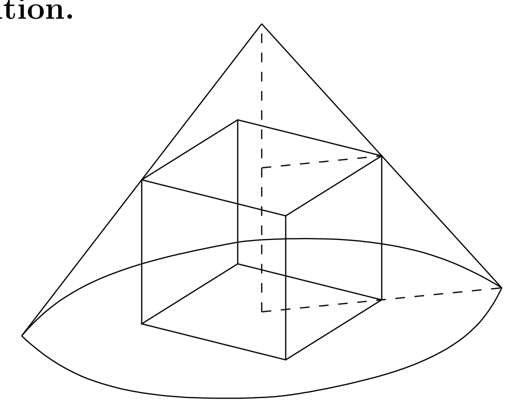


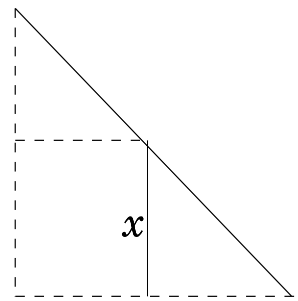
First notice that triangles CDG and CDF are congruent. So $CF = 2$, and then $BF = 4$.
The Pythagorean theorem for triangles BFD gives
$r^2 + 4^2 = BD^2$
$BE = \sqrt{r^2 + 16}$
Then the Pythagorean theorem for triangle BCG gives
$(r + \sqrt{r^2 + 16})^2 + 2^2 = 6^2$
$r^2 + (r^2 + 16) + 2r\sqrt{r^2 + 16} + 4 = 36$
$2r^2 + 2r\sqrt{r^2 + 16} = 16$
$r^2 + r\sqrt{r^2 + 16} = 8$
$r\sqrt{r^2 + 16} = 8 - r^2$
$r^2(r^2 + 16) = 64 - 16r^2 + r^4$
$r^4 + 16r^2 = 64 - 16r^2 + r^4$
$32r^2 = 64$
$r^2 = 2$
The area of the circle is $\pi r^2 = 2\pi$.
Solution 3.
$BG^2 + 2^2 = 6^2$
$BG = \sqrt{32} = 4\sqrt{2}$
Since triangles FDB and GCB are similar, $\dfrac{FD}{FB} = \dfrac{GC}{GB}$
$\frac{r}{4} = \frac{2}{4\sqrt{2}}$
$r = \sqrt{2}$
The area of the circle is $\pi r^2 = 2\pi$.

2. (MH 2014 11-12) A right circular cone has height equal to radius. What is the ratio of its volume to that of a cube inscribed inside it, with the base of the cube lying on the base of the cone?
   (a) $\frac{\pi}{12}(10 + \sqrt{2})$
   (b) $\frac{\pi}{12}(10 + 3\sqrt{2})$
   (c) $\frac{\pi}{12}(10 + 5\sqrt{2})$
   (d) $\frac{\pi}{12}(10 + 7\sqrt{2})$

(e) $\frac{\pi}{12}(10+9\sqrt{2})$

Solution.





$\frac{\sqrt{2}}{2}x + x = r = h$

$\dfrac{V_{cone}}{V_{cube}} = \dfrac{\frac{1}{3}\pi r^2 h}{x^3} = \dfrac{\frac{1}{3}\pi\left(\frac{\sqrt{2}}{2}x + x\right)^3}{x^3} = \dfrac{\frac{1}{3}\pi\left(\frac{\sqrt{2}}{2}+1\right)^3 x^3}{x^3} = \dfrac{1}{3}\pi\left(\dfrac{\sqrt{2}}{2}+1\right)^3 = \dfrac{1}{3}\pi\left(\dfrac{10}{4}+\dfrac{7\sqrt{2}}{4}\right) =$
$\dfrac{1}{12}\pi(10+7\sqrt{2})$

(d)

3. (LF 2014 9-10) Two cubes (length = width = height) have respective volumes V_1 and V_2 that satisfy $V_1/V_2 = 10$. Let S_1 and S_2 be the respective surface areas of the cubes, i.e. S_1 corresponds to V_1 and S_2 corresponds to V_2. Determine the ratio of surface areas S_1/S_2.

   (a) $S_1/S_2 = \sqrt[3]{150}$

   (b) $S_1/S_2 = \sqrt[3]{10}$

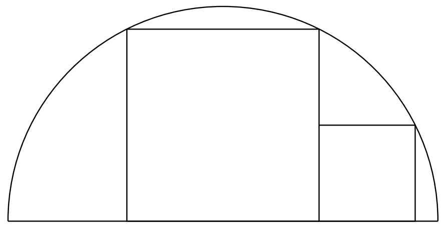
   (c) $S_1/S_2 = \sqrt[3]{200}$

   (d) $S_1/S_2 = \sqrt[3]{100}$

   (e) None of these

   Solution.
   Let x_1 and x_2 be the lengths of the edges of the two cubes. Then
   $10 = \frac{V_1}{V_2} = \frac{x_1^3}{x_2^3} = \left(\frac{x_1}{x_2}\right)^3$
   $\frac{x_1}{x_2} = \sqrt[3]{10}$
   Then
   $\frac{S_1}{S_2} = \frac{6x_1^2}{6x_2^2} = \frac{x_1^2}{x_2^2} = \left(\frac{x_1}{x_2}\right)^2 = (\sqrt[3]{10})^2 = (10^{1/3})^2 = 10^{2/3} = (10^2)^{1/3} = 100^{1/3} = \sqrt[3]{100}.$
   (d)

4. (LF 2012 9-12) In the figure below, the semicircle has radius equal to 1 inch, and the two adjacent squares are inscribed as pictured. What is the area of the smaller square?



(a) Area $= \frac{1}{4}$ in^2

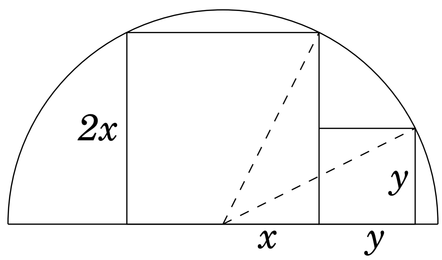
(b) Area $= \frac{1}{\sqrt{5}}$ in^2

(c) Area $= \frac{2}{\sqrt{7}}$ in^2

(d) Area $= \frac{1}{1+\sqrt{5}}$ in^2

(e) None of these

Solution.



By the Pythagorean theorem,
$x^2 + (2x)^2 = 1$ and $(x + y)^2 + y^2 = 1$.
Solving the first equation for x and then the second equation for y gives
$x = \frac{1}{\sqrt{5}}$, $y = \frac{1}{\sqrt{5}}$.
The area of the smaller square is $y^2 = \frac{1}{5}$.
(e)

5. (MH 2014 11-12) $ABCD$ is a rectangle in which the length AB minus the length AD equals 10. Inside $ABCD$ is a square $WXYZ$ with sides parallel to those of the rectangle, and W closest to A, and X closest to B. The total of the areas of the trapezoids $XBCY$ and $AWZD$ is 1000, while the total area of the trapezoids $ABXW$ and $ZYCD$ is 400. What is the area of the square $WXYZ$?

   (a) 400

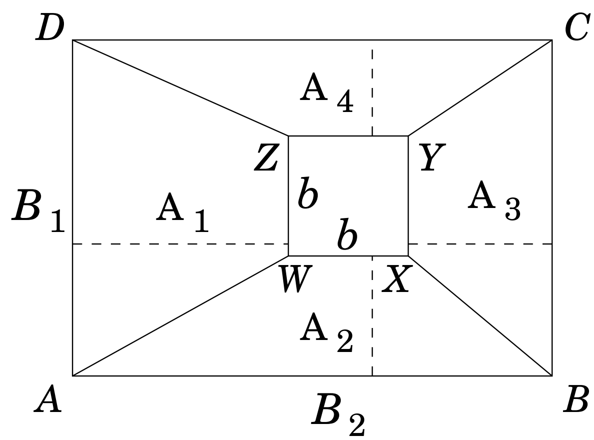
   (b) 1600

   (c) 3600

   (d) 4900

   (e) 6400

   Solution.

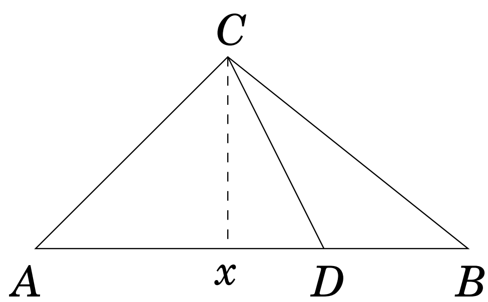


$A_1 + A_3 = 1000,$
$\frac{(B_1+b)h_1}{2} + \frac{(B_1+b)h_3}{2} = 1000$
$(B_1 + b)(h_1 + h_3) = 2000$
$(B_1 + b)(B_2 - b) = 2000$
$B_1B_2 - B_1b + B_2b - b^2 = 2000 \qquad (1)$
Similarly,
$A_2 + A_4 = 400$
$\frac{(B_2+b)h_2}{2} + \frac{(B_2+b)h_4}{2} = 400$
$(B_2 + b)(h_2 + h_4) = 800$
$(B_2 + b)(B_1 - b) = 800$
$B_1B_2 + B_1b - B_2b - b^2 = 800 \qquad (2)$
Subtracting (2) from (1) gives
$2B_2b - 2B_1b = 1200$
$B_2b - B_1b = 600$
$(B_2 - B_1)b = 600$
$10b = 600$
$b = 60$
The area of the square $WXYZ$ is $b^2 = 60^2 = 3600$.
(c)

6. (LF 2014 9-10) In $\triangle ABC$, point D lies on the side AB. The length of AB is 10 feet and the length of AD is x feet. What is the value of x such that the area enclosed by $\triangle ADC$ is twice the area enclosed by $\triangle BDC$?

   (a) $x = 8$
   (b) $x = \frac{9}{2}$
   (c) $x = 5$
   (d) $x = \frac{20}{3}$
   (e) None of these

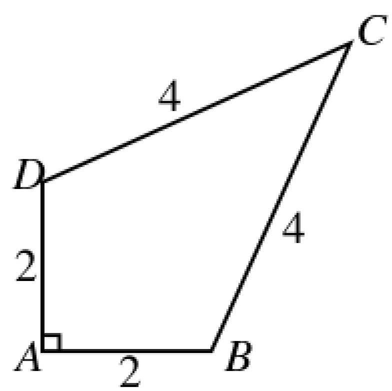
   Solution.



Triangles ADC and BDC have the same height. Since the area of $\triangle ADC$ is twice the area of $\triangle BDC$, the base AD is twice the base BD. Therefore AD must be $\frac{2}{3}$ of AB, i.e. $\frac{2}{3} \cdot 10 = \frac{20}{3}$. (d)

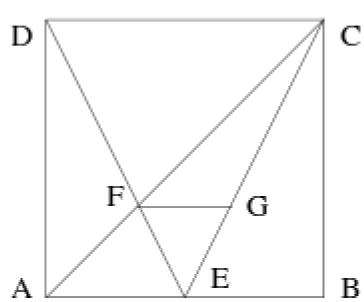
More practice problems

1. (LF 2011 9-12) In the figure below, the lengths are as labeled and the angle at A is a right angle. The area enclosed by $ABCD$ is ...



   (a) $4 + 2\sqrt{7}$
   (b) $2 + 2\sqrt{6}$
   (c) $2 + 2\sqrt{7}$
   (d) $4 + \sqrt{6}$
   (e) None of these

2. (LF 2014 11-12) The square $ABCD$ has sides of length 2. Point E is the midpoint of edge AB. Point F is the intersection of lines AC and DE. Line FG is parallel to line AB. The area of $\triangle EFG$ is:



(a) $\frac{2}{3}$

(b) $\frac{1}{3}$

(c) $\frac{2}{9}$

(d) $\frac{4}{9}$

(e) None of the above

3. (LF 2014 9-10) A cube of ice has melted so that its surface area has decreased by 19%. Assuming that at all times, the cube maintains length = width = height, by what percentage has the volume decreased?

   (a) 26.7%

   (b) 26.9%

   (c) 27.1%

   (d) 27.3%

   (e) None of these

4. (MH 2014 11-12) Points A, C, and D lie on a circle. Point B lies outside the circle such that B, D, and C are collinear with D between B and C, and BA is tangent to the circle. If $AB = 2$, $AC = 3$, and $BD = 1$, what is the area of triangle ABC?

   (a) 1

   (b) 2

   (c) $\frac{3}{4}\sqrt{15}$

   (d) $\frac{3}{4}\sqrt{11}$

   (e) $2\sqrt{11}$

5. (LF 2014 9-10) The sum of the first 2014 positive odd integers is subtracted from the sum of the first 2014 positive even integers. What is the result?

   (a) 1

   (b) 0

   (c) 4028

   (d) 2013

   (e) None of these

6. (MH 2014 11-12) Let BE be a median of triangle ABC, and let D be a point on AB such that $BD/DA = 3/7$. What is the ratio of the area of triangle BED to that of triangle ABC?

   (a) 3/20

   (b) 7/20

   (c) 1/5

   (d) 1/4

   (e) the answer cannot be determined

7. (LF 2013 11-12) One sphere is inscribed in a cube, while the cube is also inscribed in another

sphere. Find the ratio of the volumes of the larger sphere to the smaller sphere.

(a) $\sqrt{3}$

(b) $2\sqrt{3}$

(c) $3\sqrt{3}$

(d) $3\sqrt{2}$

(e) None of the above