CSU Fresno Problem Solving Session

Problem Solving Sessions website:
`http://zimmer.csufresno.edu/~mnogin/mfd-prep.html`

Math Field Day date: Saturday, April 17, 2010

Math Field Day website:
`http://www.csufresno.edu/math/news_and_events/field_day/`

Trigonometry and Geometry, 27 Feb 2010

Important formulas and theorems

Formulas for areas of basic shapes (triangle, rectangle, trapezoid, circle), perimeter, circumference, length of an arc of a circle ($l = \pi\theta$)

Similar triangles (proportional sides)

Proportional reasoning: perimeter grows proportionally to the length, area grows proportionally to the square of the length, etc.

Pythagorean theorem

Sum of angles in any triangle is $180°$, in any n-gon $(n-2)\cdot 180°$.

Ratios of lengths of sides of $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles

Trigonometric functions: definition (using triangle and unit circle), exact values for some angles, graphs, various identities

Law of sines, law of cosines

Examples

1. (MH 9-10 2006) The circumference of the front tires of a sport car is 2 feet more than the circumference of the rear tires. If the rear wheels make 440 revolutions in traveling 1 mile, find the radius of the front tires.
   (a) 4π feet
   (b) 3π feet
   (c) $\dfrac{6}{\pi}$ feet
   (d) $\dfrac{7}{\pi}$ feet

   Solution. The circumference of the rear wheels is $\dfrac{5280}{440} = 12$ ft.
   The circumference of the front tires is $12 + 2 = 14$ ft.
   The radius of the front tires is $\dfrac{14}{2\pi} = \dfrac{7}{\pi}$ ft.

2. (MH 9-10 2006) A ladder leaning against a vertical wall makes an angle of 30° with the wall. If the foot of the ladder is 3 feet from the wall, give the length of the ladder and the height it extends up the wall.

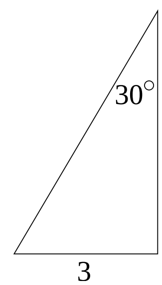
   (a) 6 feet long, $3\sqrt{3}$ feet up the wall

   (b) 1.5 feet long, $\dfrac{3\sqrt{3}}{4}$ feet up the wall

   (c) 6 feet long, $\dfrac{3\sqrt{3}}{4}$ feet up the wall

   (d) 1.5 feet long, $3\sqrt{3}$ feet up the wall

   Solution. We have a $30° - 60° - 90°$ triangle with the smaller leg (the distance from the foot of the ladder to the wall) equal to 3ft.



   The hypotenuse (length of the ladder) is 6 feet and the other leg (the height the ladder extends up the wall) is $3\sqrt{3}$ feet.

3. (MH 11-12 2006) If $\sin\alpha = \dfrac{1}{5}$ and α lies in the second quadrant, in what quadrant is 2α?

   (a) I

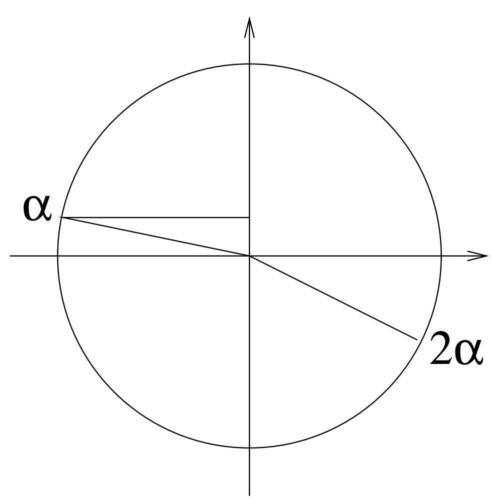
   (b) II

   (c) III

   (d) IV

   (e) Not enough information given.

   Solution. Using the definition of sin:



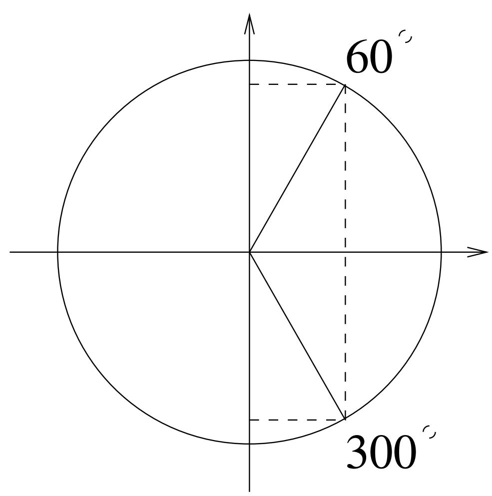
$\frac{3}{4}\pi < \alpha < \pi$, so

$\frac{6}{4}\pi < \alpha < 2\pi$

$\frac{3}{2}\pi < \alpha < 2\pi$

4. (MH 11-12 2006) Find $\tan(300°)$.
   (a) $-\sqrt{3}$
   (b) $-\frac{1}{\sqrt{3}}$
   (c) $\frac{\sqrt{3}}{2}$
   (d) 1
   (e) None of the above

   Solution. $\tan(300°) = \frac{\sin(300°)}{\cos(300°)} = \frac{-\sin(60°)}{\cos(300°)} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}.$



Mad Hatter

1. (MH 11-12 2006) The angles of a quadrilateral have degree measures that are four consecutive odd numbers. What is the degree measure of the smallest angle?
   (a) 85
   (b) 87
   (c) 88
   (d) 89
   (e) None of the above

   Solution. Let the degree measures of the four angles be:
   $2n - 3$, $2n - 1$, $2n + 1$, and $2n + 3$.

Then $(2n-3)+(2n-1)+(2n+1)+(2n+3)=360°$
$8n=360°$
$n=45°$
$2n-3=87°$.

2. (MH 9-10 2006) If an arc of 45° on circle A has the same length as an arc of 30° on circle B, then the ratio of the area of circle A to the area of circle B is

   (a) $\frac{9}{4}$

   (b) $\frac{3}{2}$

   (c) $\frac{2}{3}$

   (d) $\frac{4}{9}$

   Solution. Let r_1 be the radius of circle A, then the length of the arc of 45° is $L=r_1\cdot\frac{\pi}{4}$.
   Similarly, let r_2 be the radius of circle B, then the length of the arc of 30° is $L=r_2\cdot\frac{\pi}{6}$.
   Then $r_1=\frac{4}{\pi}L$ and $r_2=\frac{6}{\pi}L$, so
   $\frac{r_1}{r_2}=\frac{\frac{4}{\pi}L}{\frac{6}{\pi}L}=\frac{4}{6}=\frac{2}{3}$.
   Then the ratio of areas is $\left(\frac{2}{3}\right)^2=\frac{4}{9}$.

3. (MH 11-12 2006) Simplify: $\frac{\sin^4 x-\cos^4 x}{\cos x-\sin x}$

   (a) $\sin^3 x-\cos^3 x$

   (b) $\cos^3 x-\sin^3 x$

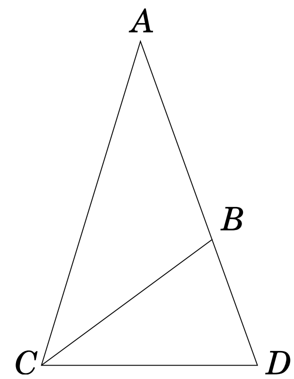
   (c) $\sin x+\cos x$

   (d) $-\cos x-\sin x$

   (e) None of the above

   Solution. $\frac{\sin^4 x-\cos^4 x}{\cos x-\sin x}=\frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)}{\cos x-\sin x}=\frac{(\sin x-\cos x)(\sin x+\cos x)(\sin^2 x+\cos^2 x)}{\cos x-\sin x}=-(\sin x+\cos x)(\sin^2 x+\cos^2 x)=-(\sin x+\cos x)$.

4. (MH 9-10 2006) Observe triangle ACD in the figure below. Segment AB is congruent to segment BC, and BC is congruent to segment CD. If segment BC bisects angle ACD, find m$\angle CAD$.



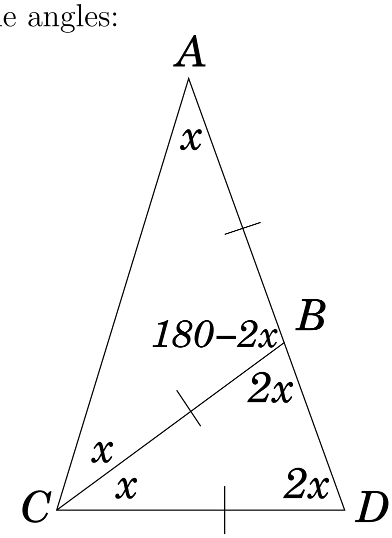
(a) 30°

(b) 36°

(c) 45°

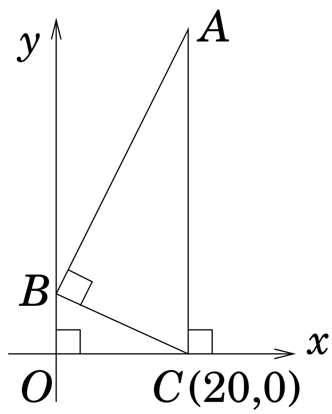
(d) 60°

Solution. Label all the angles:



The sum of all the angles in triangle ACD is $5x = 180°$, so $x = 36°$.

5. (MH 9-10 2006) In the figure below, if $\dfrac{AB}{BC} = \dfrac{4}{3}$, what is the y-coordinate of point B?

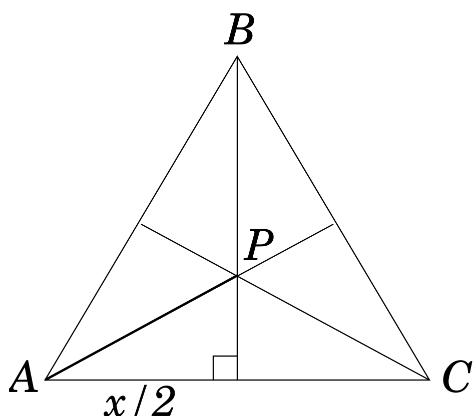


Solution. Observe that triangles ABC and COB are similar. So $\dfrac{CO}{OB} = \dfrac{AB}{BC} = \dfrac{4}{3}$.

Since $CO = 20$, it follows that $OB = 15$.

6. (MH 9-10 2006) Let ABC be an equilateral triangle with sides of length x. Let P be the point of intersection of the three angle bisectors. Find the length of AP.
   (a) $\dfrac{x\sqrt{3}}{3}$
   (b) $\dfrac{x\sqrt{3}}{6}$
   (c) $\dfrac{5x\sqrt{3}}{6}$
   (d) $\dfrac{2x\sqrt{3}}{3}$

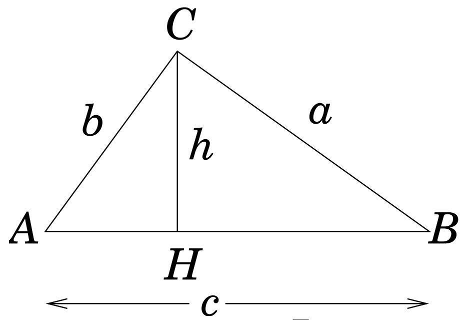
   Solution. Using a $30° - 60° - 90°$ triangle:
   $\dfrac{AP}{x/2} = \dfrac{2}{\sqrt{3}}$, so
   $AP = \dfrac{x}{\sqrt{3}} = \dfrac{x\sqrt{3}}{3}$.



Leap Frog

1. (MH 11-12 2006) The hypotenuse AB of a right triangle ABC is 6 and the height CH is $\dfrac{4\sqrt{2}}{3}$. Find the smaller leg of the triangle.
   (a) 2
   (b) 3
   (c) $2\sqrt{2}$
   (d) $2\sqrt{3}$
   (e) None of the above

   Solution. The Pythagorean theorem gives $a^2 + b^2 = c^2$ and from similar triangles ABC and ACH we have $\dfrac{c}{a} = \dfrac{b}{h}$, which implies $ab = ch$, so:
   $a^2 + b^2 = 36$ and $ab = 8\sqrt{2}$



Solving the second equation for b, $b = \dfrac{8\sqrt{2}}{a}$.
Substituting this into the first equation, we get
$a^2 + \left(\dfrac{8\sqrt{2}}{a}\right)^2 = 36$
$a^2 + \dfrac{128}{a^2} = 36$
$a^4 + 128 = 36a^2$
$a^4 - 36a^2 + 128 = 0$
Let $x = a^2$, then $x^2 - 36x + 128 = 0$
$(x-4)(x-32) = 0$
$a^2 = 4$, $a^2 = 32$
$a = 2$, $a = 4\sqrt{2}$.
The smaller solution is $a = 2$.

2. (MH 11-12 2006) The diagonals of a trapezoid divide it into four triangles. Find the area of the trapezoid if the areas of the two triangles adjacent to the bases of the trapezoid are S_1 and S_2.

   (a) $S_1 + S_2$

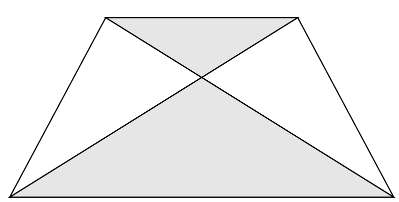
   (b) $S_1 \cdot S_2$

   (c) $S_1 + S_2 + S_1 \cdot S_2$

   (d) $\sqrt{S_1} \cdot \sqrt{S_2}$

   (e) $\left(\sqrt{S_1} + \sqrt{S_2}\right)^2$

   Solution. Let S be the area of the trapezoid.



(a) is obviously wrong because $S > S_1 + S_2$.
(b) is wrong because if the linear size of the trapezoid is increased by a factor of 2, then both S_1 and S_2 will increase by a factor of 4, and so should S, but $S_1 \cdot S_2$ would increase by a factor of 16. Alternatively, all of S_1, S_2, and S are measured in square units, but $S_1 \cdot S_2$ would have units to the 4th power.

(c) is wrong for a reason similar to that for (b): the units are wrong.
(d): consider the special case when $S_1 = S_2 = 1$, then $\sqrt{S_1} \cdot \sqrt{S_2} = 1$, but S must be larger than 2.
The only choice left is (e), which can be shown to be a correct formula.

3. (MH 9-10 2006) If, in a triangle ABC, median BD is such that $\text{m}\angle A = \text{m}\angle DBC$, and $\text{m}\angle ADB = 45°$, find $\text{m}\angle A$.

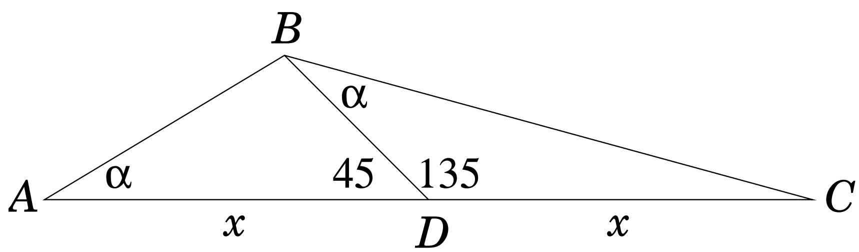
   (a) 30°

   (b) 45°

   (c) 60°

   (d) none of the above

   Solution.

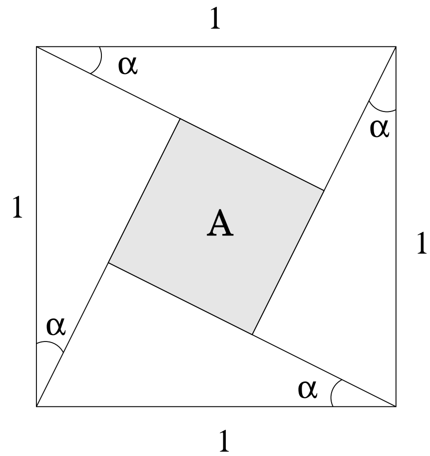
   

   Observe that triangles ABC and BDC are similar. Therefore $\frac{BC}{AC} = \frac{DC}{BC}$.
   Let $AD = DC = x$, then $AC = 2x$, so
   $\frac{BC}{2x} = \frac{x}{BC}$

   $BC^2 = 2x^2$
   $BC = \sqrt{2}x$.

   Using the Law of Sines for triangle ABC,
   $\frac{AC}{\sin \angle ABC} = \frac{BC}{\angle A}$
   $\frac{2x}{\sin(135°)} = \frac{\sqrt{2}x}{\sin \alpha}$
   $\frac{2}{\sqrt{2}/2} = \frac{\sqrt{2}}{\sin \alpha}$
   $\frac{4}{\sqrt{2}} = \frac{\sqrt{2}}{\sin \alpha}$
   $\sin \alpha = \frac{\sqrt{2} \cdot \sqrt{2}}{4}$
   $\sin \alpha = \frac{1}{2}$
   $\alpha = 30°$.

4. (LF 9-12 2006) The area of the small square in the figure below is $A =$ _____ (square units).



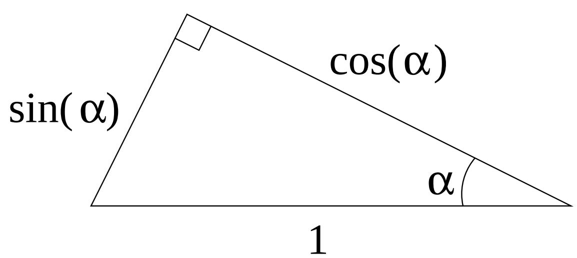
(a) $\cos^2(2\alpha)$

(b) $\cos(2\alpha)$

(c) $1 - \sin^2(2\alpha°)$

(d) $1 - \sin(2\alpha°)$

Solution. The region inside the large square but outside the small square is seen to be the sum of 4 mutually congruent right triangles. One of the triangles is pictured below.

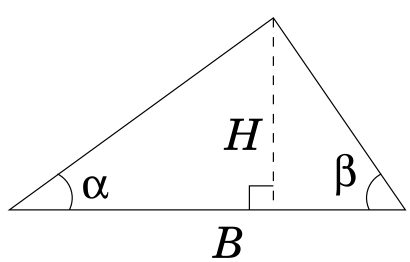


The area of this triangle is $\frac{1}{2}\sin(\alpha°)\cos(\alpha°) = \frac{1}{4}\sin(2\alpha°)$ square units. Since the 4 triangles have equal area, the area of the small square is

$$1 - 4 \times \frac{1}{4}\sin(2\alpha°) = 1 - \sin(2\alpha°)$$

square units.

5. (LF 9-12 2005) The height H of the triangle below, as a function of angle measures α, β, and base length B, is _____.



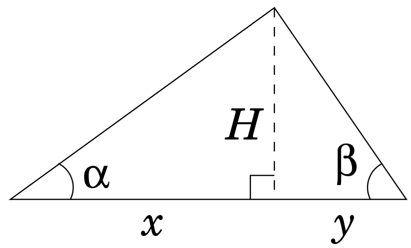
(a) $B(\tan\alpha + \tan\beta)$

(b) $B(\cot\alpha + \cot\beta)$

(c) $\dfrac{B}{\cot\alpha + \cot\beta}$

(d) $\dfrac{B}{\tan\alpha + \tan\beta}$

Solution. The base naturally splits into two pieces, $B = x + y$, as pictured.



First note that $\tan\alpha = H/x$ and $\tan\beta = H/y$. From here, we can solve for $x = H\cot\alpha$ and $y = H\cot\beta$. Thus

$$B = x + y = H(\cot\alpha + \cot\beta).$$

Solve for H from this last equation.

$$H = \frac{B}{\cot\alpha + \cot\beta}.$$

6. (LF 9-12 2006) Let x be a solution to the equation $\sin(x)\cos(2x) = -\sin(2x)\cos(x)$ for which $0 < x < \pi/2$ (in radians). Then $\tan(x) =$ _____.

   (a) $\dfrac{\sqrt{3}}{2}$

   (b) $\sqrt{3}$

   (c) $2\sqrt{3}$

   (d) $\dfrac{2}{\sqrt{3}}$

Solution. The double angle identities allow us to re-write the equation as

$$\sin(x)\left(\cos^2(x) - \sin^2(x)\right) = -2\sin(x)\cos(x)\cos(x).$$

Since $\sin(x) \neq 0$, we may divide out this term:

$$\cos^2(x) - \sin^2(x) = -2\cos(x)\cos(x).$$

A little more rearrangement gives

$$3\cos^2(x) = \sin^2(x).$$

We may divide by $\cos^2(x)$ (which also cannot be equal to zero) to get

$$\tan^2(x) = 3$$

and so $\tan(x) = \sqrt{3}$. (The negative square root can be ignored because $0 < x < \pi/2$.)

More Problems

1. (MH 11-12 2006) ABC is a right triangle, C is its right angle, CH is the corresponding height, and CD is the corresponding bisector. The angle between CH and CD is $26°$. Find the acute angles of the triangle ABC.

   (a) $13°$, $77°$

   (b) $19°$, $71°$

   (c) $26°$, $64°$

   (d) $38°$, $52°$

   (e) None of the above

   Answer. (b) Hint: find all angles in this triangle.

2. (MH 11-12 2006) The bases of an isosceles trapezoid are 22 cm and 18 cm, and the height is 10 cm. Find the radius of the circle circumscribed around the trapezoid.

   (a) 10 cm

   (b) 11 cm

   (c) 12 cm

   (d) $2\sqrt{33}$ cm

   (e) $\sqrt{130}$ cm

   Answer. (e) Hint: draw a picture, draw a height of the trapezoid passing through the center of the circle. Let the center divide the height into pieces x and $10 - x$. Draw radii to the vertices of the trapezoid. Use the Pythagorean theorem to write an equation with x.

3. (MH 11-12 2006) The circumference of the front tires of a car is 25 cm less than the circumference of the rear tires. If the front wheels make 800 revolutions in traveling 1 km, find the radius of the rear tires.

   (a) $\dfrac{75}{\pi}$ cm

   (b) $\dfrac{90}{\pi}$ cm

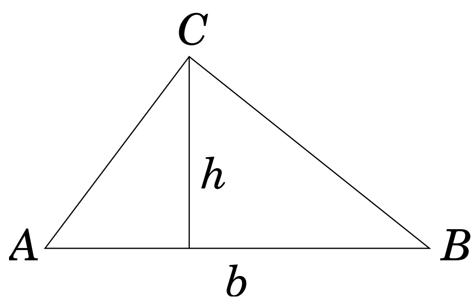
   (c) 25π cm

   (d) 30π cm

   (e) None of the above

   Answer. (a) Hint: this problem is similar to problem 1 on page 1.

4. (LF 9-12 2006) A formula for the height h in terms of the base length b and the two base angles $\angle A$ and $\angle B$ for the triangle pictured below is _____.



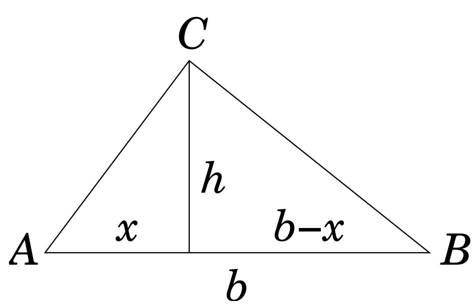
(a) $\dfrac{b}{\cot\angle A + \cot\angle B}$

(b) $\dfrac{b}{\sqrt{\cot^2\angle A + \cot^2\angle B}}$

(c) $\dfrac{b}{1 + \cot\angle A\cot\angle B}$

(d) $\dfrac{b}{\sqrt{1 + \cot^2\angle A\cot^2\angle B}}$

Solution. Add the labels x and $b - x$ as the obvious lengths in the picture below.



From the picture we can see that $\tan\angle A = \dfrac{h}{x}$ and $\tan\angle B = \dfrac{h}{b-x}$. These two formulas can be rewritten as

$$x = h\cot\angle A \text{ and } b - x = h\cot\angle B.$$

Adding these two formulas together gives

$$b = h\cot\angle A + h\cot\angle B.$$

Solve for h to get

$$h = \frac{b}{\cot\angle A + \cot\angle B}.$$