The Logic of a Topological Space

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Outline

1 Preliminaries

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- Set operations and logical connectives

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   - Results and open questions
   - Applications

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$$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R) \qquad P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

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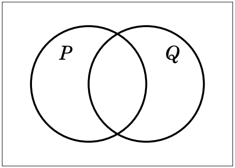

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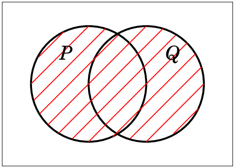

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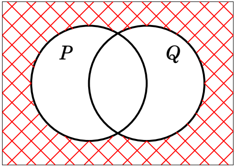

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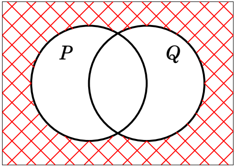

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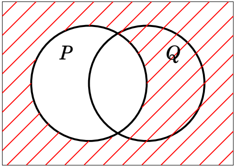

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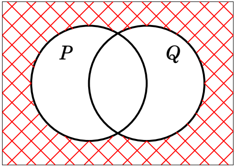

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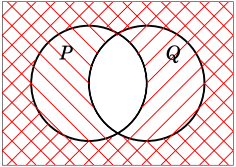

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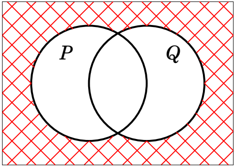

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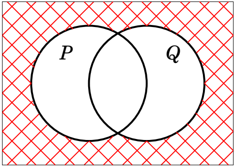

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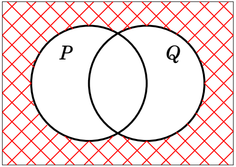
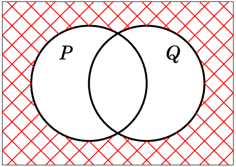


$\overline{P \cup Q}$

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T



$\overline{P} \cap \overline{Q}$

P	Q	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
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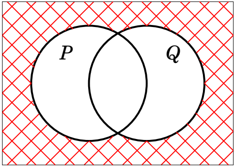

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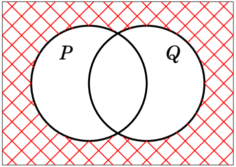

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T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Axioms

1. $(P \wedge Q) \to P$
2. $(Q \wedge P) \to P$
3. $P \to (P \vee Q)$
4. $P \to (Q \vee P)$
5. $\neg\neg P \to P$
6. $P \to (Q \to P)$
7. $P \to (Q \to (P \wedge Q))$
8. $\big((P \to Q) \wedge (P \to \neg Q)\big) \to \neg P$
9. $\big((P \to R) \wedge (Q \to R)\big) \to \big((P \vee Q) \to R\big)$
10. $\big((P \to Q) \wedge (P \to (Q \to R))\big) \to (P \to R)$

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10. $\big((P \to Q) \wedge (P \to (Q \to R))\big) \to (P \to R)$

Rule of inference

$$\frac{P, \; P \to Q}{Q}$$

Example: derive $(A \vee B) \to (B \vee A)$

1. Axiom $P \to (P \vee Q)$: $B \to (B \vee A)$
2. Axiom $P \to (Q \vee P)$: $A \to (B \vee A)$
3. Axiom $P \to \big(Q \to (P \wedge Q)\big)$:
   $(A \to B \vee A) \to \Big((B \to B \vee A) \to ((A \to B \vee A) \wedge (B \to B \vee A))\Big)$
4. Steps 2 and 3:
   $(B \to B \vee A) \to ((A \to B \vee A) \wedge (B \to B \vee A))$
5. Steps 1 and 4: $(A \to B \vee A) \wedge (B \to B \vee A)$
6. Axiom $\Big((P \to R) \wedge (Q \to R)\Big) \to \Big((P \vee Q) \to R\Big)$:
   $\Big((A \to B \vee A) \wedge (B \to B \vee A)\Big) \to \Big((A \vee B) \to (B \vee A)\Big)$
7. Steps 5 and 6: $(A \vee B) \to (B \vee A)$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
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Given a mapping from propositional variables (P, Q, *etc.*) to subsets of X, every formula is mapped to a subset X.

e.g.

$$P \wedge Q \mapsto P \cap Q$$
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Some formulas are always mapped to the whole set X. They are called valid with respect to interpretation in X.

Soundness and completeness

Theorem. Let X be a set.

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The language of classical logic does not distinguish different non-empty sets X.

Topological spaces

Definition. A topological space is a set X together with a collection of subsets of X, called open subsets, satisfying the following axioms:

- The empty subset and X are open.
- The union of any collection of open subsets is also open.
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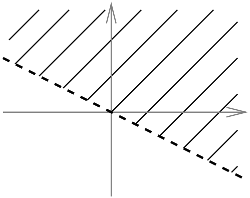
Example. $X = \mathbb{R}^n$. A subset P of X is open iff for any point x in P, some open ball containing x is contained in P.

$\mathbb{R}$:

$\mathbb{R}^2$:



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Definition. Let X and Y be topological spaces. Then $f\colon X \to Y$ is continuous if for any open subset U of Y, $f^{-1}(U)$ is an open subset of X.

Quantifiers

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Example. Let P be a subset of $\mathbb{R}^2$. Then

$$\forall x \in P\ \exists r \in \mathbb{R}\ \Big((r > 0) \wedge \forall y \in \mathbb{R}^2 \big(\mathrm{dist}(x, y) < r \rightarrow y \in P\big)\Big)$$

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The language with quantifiers is very expressive but undecidable.

Compromise: modality

The classical logic is extended with an operator $\Box$.
Interpretations of $\Box P$:

- P is known
- P is provable
- P is computable
- P is necessary
- P will always be true
- P will be true tomorrow
- *etc.*

S4: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\Box$

- Axioms of classical logic
- $\Box P \rightarrow P$
- $\Box P \rightarrow \Box\Box P$
- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

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Rules of inference

$$\frac{P, \;\; P \rightarrow Q}{Q} \quad \text{and} \quad \frac{P}{\Box P}$$

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Topological interpretation of $\Box$:
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Theorem. Let X be a topological space. Then S4 is sound with respect to interpretation in X.

Theorem. S4 is complete with respect to all interpretations in all topological spaces X, i.e. for any formula F, the following statements are equivalent:

1. F is derivable in S4
2. F is valid in each interpretation (for each topological space X)

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Corollary. The modal logic (with operations $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\Box$) does not distinguish $\mathbb{R}^n$'s for different n.

Problem

Start with a subset S of $\mathbb{R}$.

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S

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S
$\text{inter}(S)$

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Can there be infinitely many different sets in these sequences?

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⋮

compl(S)
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⋮

Can there be infinitely many different sets in these sequences?

If not, what is the maximum number of different sets?

Example 1

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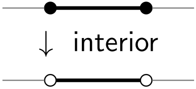

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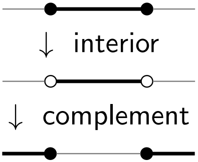

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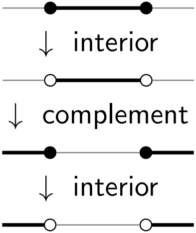

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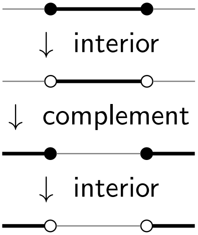
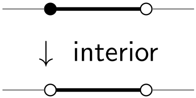


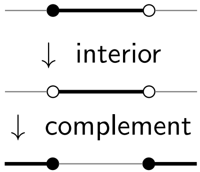
Get 4 different subsets of $\mathbb{R}$

Example 2

Example 2



Example 2



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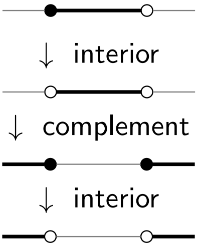

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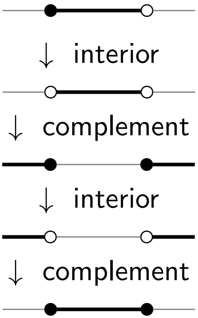

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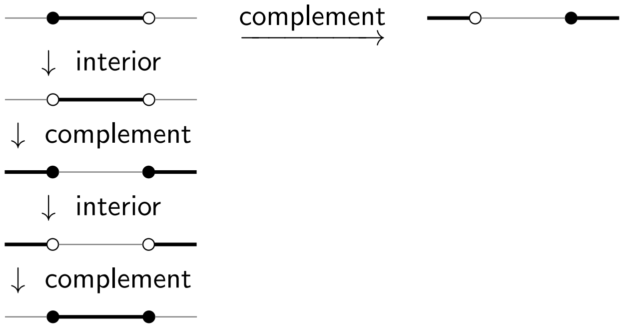

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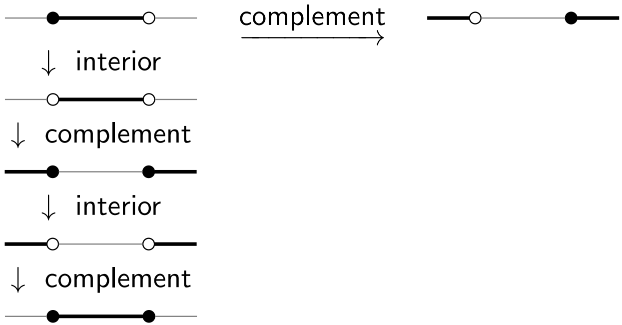
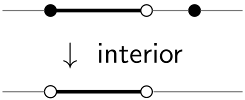


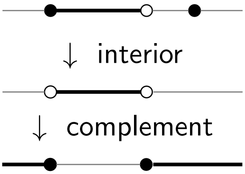
Get 6 different subsets of $\mathbb{R}$

Example 3

Example 3



Example 3



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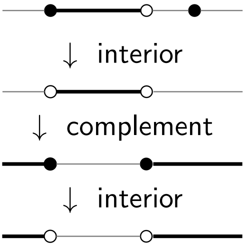

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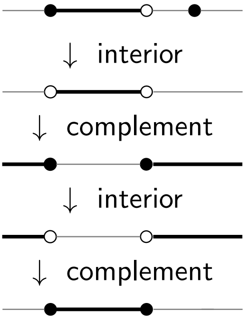

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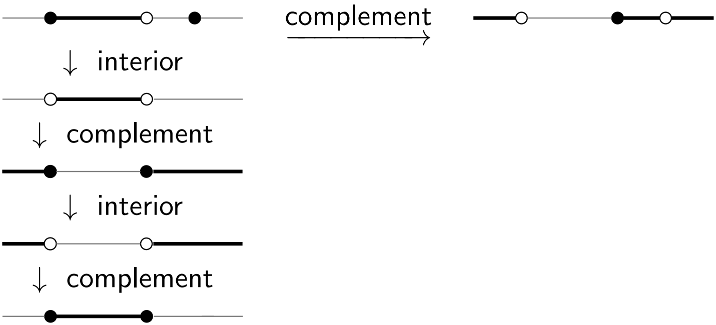

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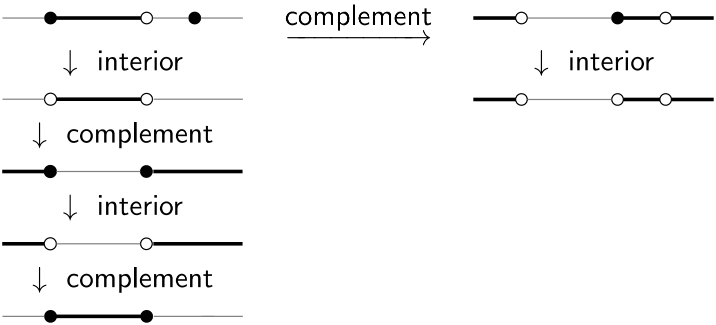
complement
interior
complement
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interior

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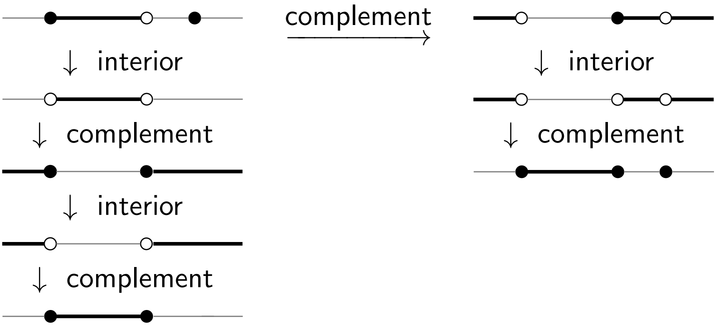

complement
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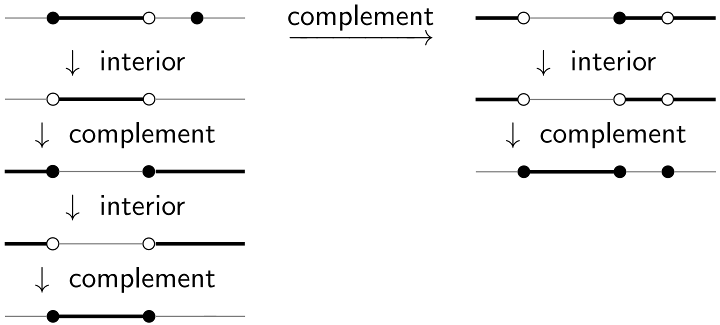
Example 3



Get 8 different subsets of $\mathbb{R}$

Problem

Can there be infinitely many different sets?

Problem

Can there be infinitely many different sets?

Answer: No.

Problem

Can there be infinitely many different sets?
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What is the largest possible number of different sets?

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Proof that we cannot get more than 14.

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What is the largest possible number of different sets?
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Proof that we cannot get more than 14.

Lemma. There are at most 7 different sets in the sequence

S
$\text{inter}(S)$
$\text{compl}(\text{inter}(S))$
$\text{inter}(\text{compl}(\text{inter}(S)))$
$\vdots$

because

$\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(S))))))) =$
$\text{inter}(\text{compl}(\text{inter}(S)))$.

Proof

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

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Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

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Notation: $\Diamond R \equiv \neg\Box\neg R$.

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In the topological interpretation "$\Diamond R$" means "the closure of R".

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Want to prove: $\Box\Diamond\Box\Diamond T \equiv \Box\Diamond T$.

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Proof of $\Box\Diamond T \to \Box\Diamond\Box\Diamond T$. Axiom: $\Box P \to P$

Let $P = \neg R$, then $\Box\neg R \to \neg R$

Contrapositive: $R \to \neg\Box\neg R$

Let $R = \Box Q$, then $\Box Q \to \neg\Box\neg\Box Q$

i.e. $\Box Q \to \Diamond\Box Q$

Apply $\Box$: $\Box\Box Q \to \Box\Diamond\Box Q$

Axiom: $\Box Q \to \Box\Box Q$

Therefore $\Box Q \to \Box\Diamond\Box Q$

Let $Q = \Diamond T$, then $\Box\Diamond T \to \Box\Diamond\Box\Diamond T$.

Proof

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

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Apply $\Box$: $\Box\Box Q \to \Box\Diamond\Box Q$

Axiom: $\Box Q \to \Box\Box Q$

Therefore $\Box Q \to \Box\Diamond\Box Q$

Let $Q = \Diamond T$, then $\Box\Diamond T \to \Box\Diamond\Box\Diamond T$.

Similarly $\Box\Diamond\Box\Diamond T \to \Box\Diamond T$.

Proof

Similarly, there are at most 7 different subsets in the sequence

$$\begin{array}{l} \text{compl}(S) \\ \text{inter}(\text{compl}(S)) \\ \text{compl}(\text{inter}(\text{compl}(S))) \\ \text{inter}(\text{compl}(\text{inter}(\text{compl}(S)))) \\ \vdots \end{array}$$

because

$\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(\text{compl}(S)))))))) = \text{inter}(\text{compl}(\text{inter}(\text{compl}(S))))$,

so at most 14 different subsets total.

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inter(compl(S))
compl(inter(compl(S)))
inter(compl(inter(compl(S))))
$\vdots$

because

inter(compl(inter(compl(inter(compl(inter(compl(S)))))))) $=$ inter(compl(inter(compl(S)))),

so at most 14 different subsets total.

Homework problem. Find a subset of $\mathbb{R}$ for which you get 14 different subsets.

Dynamic topological systems

Definition. A dynamic topological system is a topological space X with a continuous function $f\colon X \to X$.

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New modal operator $\bigcirc$: $\bigcirc P$ is interpreted as $f^{-1}(P)$.

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S4C

- Axioms of classical logic
- $\Box P \to P$
- $\Box P \to \Box\Box P$
- $\Box(P \to Q) \to (\Box P \to \Box Q)$
- $\bigcirc(P \to Q) \to (\bigcirc P \to \bigcirc Q)$
- $(\bigcirc\neg P) \leftrightarrow (\neg \bigcirc P)$
- $(\bigcirc\Box P) \leftrightarrow (\Box \bigcirc \Box P)$

Rules of inference

(1) $\dfrac{P,\ P \to Q}{Q}$

(2) $\dfrac{P}{\Box P}$ (3) $\dfrac{P}{\bigcirc P}$

Theorem. Let F be a formula. The following are equivalent:

1. F is derivable in S4C
2. F is valid with respect to every interpretation in every topological space
3. F is valid with respect to every interpretation in every $\mathbb{R}^n$

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However, the above statements are not equivalent to

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Namely, there exists a formula that is valid in $\mathbb{R}$ but not valid in any $\mathbb{R}^n$ with $n > 1$.

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However, the above statements are not equivalent to

4. F is valid with respect to every interpretation in $\mathbb{R}$

Namely, there exists a formula that is valid in $\mathbb{R}$ but not valid in any $\mathbb{R}^n$ with $n > 1$.

Corollary. The language of S4C distinguishes $\mathbb{R}$ from $\mathbb{R}^n$ for $n > 1$.

Example

Let $U = \Box P \quad (U \text{ is open}),$
$\Phi = (\Diamond U) \wedge (\Diamond \neg U) \quad (\Phi \text{ is the boundary of } U),$

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Lemma. There exist subsets P and Q of $\mathbb{R}^2$ and a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi \neq \emptyset$, i.e. $\neg\Psi \neq \mathbb{R}^2$.

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Corollary. The formula $\neg\Psi$ is not derivable in S4C.

Theorem

(joint work with A. Nogin; also by D.F. Duque)

For any $n \geq 2$, S4C is complete with respect to any interpretation in $\mathbb{R}^n$.

Dimension 1

(joint work with A. Nogin)

The following formulas are valid with respect to any interpretation in $\mathbb{R}$:

$$\bigcirc Q \wedge \Diamond(\bigcirc\neg Q \wedge \bigcirc\Diamond\neg P \wedge \Box \bigcirc P) \rightarrow \Diamond(\bigcirc\neg Q \wedge \Diamond \bigcirc \neg P \wedge \Diamond\Box \bigcirc P)$$

$$\bigcirc\neg P \wedge \bigcirc\neg Q \wedge \Diamond\Box \bigcirc P \wedge \Diamond \bigcirc (\neg P \wedge Q) \wedge \Box \bigcirc S \rightarrow \Diamond(\Diamond\Box \bigcirc P \wedge \Diamond \bigcirc \neg P \wedge \bigcirc\Box S)$$

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(joint work with A. Nogin)

The following formulas are valid with respect to any interpretation in $\mathbb{R}$:

$$\bigcirc Q \wedge \Diamond(\bigcirc\neg Q \wedge \bigcirc\Diamond\neg P \wedge \Box \bigcirc P) \to \Diamond(\bigcirc\neg Q \wedge \Diamond \bigcirc \neg P \wedge \Diamond\Box \bigcirc P)$$

$$\bigcirc\neg P \wedge \bigcirc\neg Q \wedge \Diamond\Box \bigcirc P \wedge \Diamond \bigcirc (\neg P \wedge Q) \wedge \Box \bigcirc S \to \Diamond(\Diamond\Box \bigcirc P \wedge \Diamond \bigcirc \neg P \wedge \bigcirc\Box S)$$

Open question

What exactly is the dynamic topological logic of $\mathbb{R}$?

Application: Hybrid Control Systems

- "Discrete" parameters: Discrete Mathematics
- "Continuous" parameters: Optimal Control Theory: Differential Equations, PDEs, *etc*
- Parameters of both types: Hybrid Control System: Modal Logic

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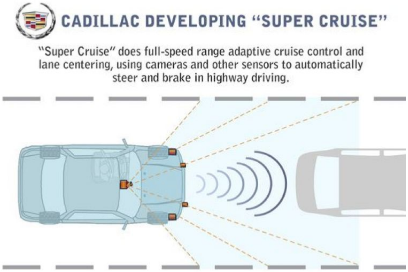

Thank you!