The Logic of a Topological Space

Maria Nogin
CSU Fresno
mnogin@csufresno.edu
1 Preliminaries
Outline

1 Preliminaries
   - Set operations and logical connectives
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   - Set operations and logical connectives
   - Topological spaces
Outline

1 Preliminaries
   ■ Set operations and logical connectives
   ■ Topological spaces

2 Modal logics
Outline

1. Preliminaries
   - Set operations and logical connectives
   - Topological spaces

2. Modal logics
   - New operator and axioms
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2 Modal logics
   - New operator and axioms
   - Topological interpretation: “interior”
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   - Topological interpretation: “interior”
   - Problem
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2 Modal logics
   - New operator and axioms
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3 Dynamic topological systems
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   - Topological interpretation: “interior”
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3 Dynamic topological systems
   - Definition
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   - Problem

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   - “Preimage” operator and new axioms
   - Results and open questions
   - Applications
Set operations and logical connectives

\[
P \cup Q = P \cap Q
\]

\[
\neg (P \lor Q) \equiv (\neg P) \land (\neg Q)
\]

\[
P = P \neg \neg P \equiv P
\]

\[
P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)
\]

\[
P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)
\]
Set operations and logical connectives

\[ P \cup Q = \overline{P} \cap \overline{Q} \quad \text{and} \quad \neg (P \lor Q) \equiv (\neg P) \land (\neg Q) \]
Set operations and logical connectives

\[ P \cup Q = \overline{P} \cap \overline{Q} \]

\[ \neg(P \lor Q) \equiv (\neg P) \land (\neg Q) \]

\[ \overline{\overline{P}} = P \]

\[ \neg\neg P \equiv P \]
Set operations and logical connectives

\[
\overline{P \cup Q} = \overline{P} \cap \overline{Q}
\]
\[
\neg(P \lor Q) \equiv (\neg P) \land (\neg Q)
\]
\[
\overline{P} = P
\]
\[
\neg\neg P \equiv P
\]
\[
P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)
\]
\[
P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)
\]
Set operations and logical connectives

\[ P \cup Q = \overline{P \cap Q} \quad \neg(P \lor Q) \equiv (\neg P) \land (\neg Q) \]
Set operations and logical connectives

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Set operations and logical connectives

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Set operations and logical connectives

\[ \overline{P \cup Q} = \overline{P} \cap \overline{Q} \]

\[ \neg(P \lor Q) \equiv (\neg P) \land (\neg Q) \]

\[ P \cap Q \]

\[ P \cup Q \]
Set operations and logical connectives

$$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$$  \hspace{1cm}  \neg(P \lor Q) \equiv (\neg P) \land (\neg Q)$$
Set operations and logical connectives

\[ \overline{P \cup Q} = \overline{P} \cap \overline{Q} \]

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Set operations and logical connectives

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Set operations and logical connectives

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\[ \neg(P \lor Q) \equiv (\neg P) \land (\neg Q) \]

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Set operations and logical connectives

\[ P \cup Q = \overline{P} \cap \overline{Q} \quad \neg(P \lor Q) \equiv (\neg P) \land (\neg Q) \]

\[ \begin{array}{c|c|c|c}
  P & Q & P \lor Q & \neg(P \lor Q) \\
  \hline
  T & T & T & F \\
  T & F & T & F \\
  F & T & T & F \\
  F & F & F & T \\
\end{array} \]

\[ \begin{array}{c|c|c|c|c|c}
  P & Q & \neg P & \neg Q & (\neg P) \land (\neg Q) \\
  \hline
  T & T & F & F & F \\
  T & F & F & T & F \\
  F & T & T & F & F \\
  F & F & T & T & T \\
\end{array} \]
Axioms

1. \((P \land Q) \rightarrow P\)
2. \((Q \land P) \rightarrow P\)
3. \(P \rightarrow (P \lor Q)\)
4. \(P \rightarrow (Q \lor P)\)
5. \(\neg P \rightarrow P\)
6. \(P \rightarrow (Q \rightarrow P)\)
7. \(P \rightarrow (Q \rightarrow (P \land Q))\)
8. \((P \rightarrow Q) \land (P \rightarrow \neg Q)\) \rightarrow \neg P
9. \((P \rightarrow R) \land (Q \rightarrow R)\) \rightarrow ((P \lor Q) \rightarrow R)
10. \((P \rightarrow Q) \land (P \rightarrow (Q \rightarrow R))\) \rightarrow (P \rightarrow R)
Axioms

1. \((P \land Q) \rightarrow P\)
2. \((Q \land P) \rightarrow P\)
3. \(P \rightarrow (P \lor Q)\)
4. \(P \rightarrow (Q \lor P)\)
5. \(\neg\neg P \rightarrow P\)
6. \(P \rightarrow (Q \rightarrow P)\)
7. \(P \rightarrow (Q \rightarrow (P \land Q))\)
8. \(\left[ (P \rightarrow Q) \land (P \rightarrow \neg Q) \right] \rightarrow \neg P\)
9. \(\left[ (P \rightarrow R) \land (Q \rightarrow R) \right] \rightarrow \left( (P \lor Q) \rightarrow R \right)\)
10. \(\left[ (P \rightarrow Q) \land (P \rightarrow (Q \rightarrow R)) \right] \rightarrow (P \rightarrow R)\)

Rule of inference

\[\begin{array}{c}
P, P \rightarrow Q \\
\hline
Q
\end{array}\]
Example: derive \((A \vee B) \to (B \vee A)\)

1. Axiom \(P \to (P \vee Q)\): \(B \to (B \vee A)\)
2. Axiom \(P \to (Q \vee P)\): \(A \to (B \vee A)\)
3. Axiom \(P \to (Q \to (P \land Q))\):
   \((A \to B \vee A) \to ((B \to B \vee A) \to ((A \to B \vee A) \land (B \to B \vee A)))\)

4. Steps 2 and 3:
   \((B \to B \vee A) \to ((A \to B \vee A) \land (B \to B \vee A))\)

5. Steps 1 and 4:
   \((A \to B \vee A) \land (B \to B \vee A)\)

6. Axiom \(\left( (P \to R) \land (Q \to R) \right) \to \left( (P \vee Q) \to R \right)\):
   \(\left( (A \to B \vee A) \land (B \to B \vee A) \right) \to \left( (A \vee B) \to (B \vee A) \right)\)

7. Steps 5 and 6:
   \((A \vee B) \to (B \vee A)\)
Let $X$ be a set.
Logical connectives are interpreted as operations on subsets of $X$:

- conjunction $\land$ – as intersection $\cap$
- disjunction $\lor$ – as union $\cup$
- negation $\neg$ – as complement $\subseteq$

Given a mapping from propositional variables ($P$, $Q$, etc.) to subsets of $X$, every formula is mapped to a subset $X$. For example:

- $P \land Q \mapsto P \cap Q$
- $P \lor \neg P \mapsto P \cup P = X$

Some formulas are always mapped to the whole set $X$. They are called valid with respect to interpretation in $X$. 
Let $X$ be a set. Logical connectives are interpreted as operations on subsets of $X$:

- **conjunction** $\land$ – as intersection $\cap$
- **disjunction** $\lor$ – as union $\cup$
- **negation** $\neg$ – as complement $\overline{\ }$

$(P \rightarrow Q) \equiv (\neg P \lor Q)$
Let $X$ be a set. Logical connectives are interpreted as operations on subsets of $X$:

- **conjunction** $\land$ – as intersection $\cap$
- **disjunction** $\lor$ – as union $\cup$
- **negation** $\neg$ – as complement $\overline{\cdot}$

$$(P \rightarrow Q) \equiv ((\neg P) \lor Q), \quad (P \leftrightarrow Q) \equiv ((P \rightarrow Q) \land (Q \rightarrow P))$$
Let $X$ be a set.
Logical connectives are interpreted as operations on subsets of $X$: 

- conjunction $\wedge$ – as intersection $\cap$ 
- disjunction $\vee$ – as union $\cup$ 
- negation $\neg$ – as complement 

$$(P \rightarrow Q) \equiv (\neg P \vee Q), \quad (P \leftrightarrow Q) \equiv ((P \rightarrow Q) \wedge (Q \rightarrow P))$$

Given a mapping from propositional variables ($P$, $Q$, etc.) to subsets of $X$, every formula is mapped to a subset $X$.

E.g. 

\[
\begin{align*}
P \wedge Q & \mapsto P \cap Q \\
P \vee \neg P & \mapsto P \cup \overline{P}
\end{align*}
\]
Let $X$ be a set. Logical connectives are interpreted as operations on subsets of $X$:

- conjunction $\land$ – as intersection $\cap$
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$(P \rightarrow Q) \equiv ((\neg P) \lor Q)$, $(P \leftrightarrow Q) \equiv ((P \rightarrow Q) \land (Q \rightarrow P))$

Given a mapping from propositional variables ($P$, $Q$, etc.) to subsets of $X$, every formula is mapped to a subset $X$.

e.g. $P \land Q \mapsto P \cap Q$
$P \lor \neg P \mapsto P \cup \overline{P} = X$
Subset interpretation

Let $X$ be a set.
Logical connectives are interpreted as operations on subsets of $X$:

- **conjunction** $\land$ – as intersection $\cap$
- **disjunction** $\lor$ – as union $\cup$
- **negation** $\neg$ – as complement $\overline{\cdot}$

$(P \rightarrow Q) \equiv ((\neg P) \lor Q)$, $(P \leftrightarrow Q) \equiv ((P \rightarrow Q) \land (Q \rightarrow P))$

Given a mapping from propositional variables ($P$, $Q$, *etc.*.) to subsets of $X$, every formula is mapped to a subset $X$.

**e.g.**

- $P \land Q \mapsto P \cap Q$
- $P \lor \neg P \mapsto P \cup \overline{P} = X$

Some formulas are always mapped to the whole set $X$. They are called **valid with respect to interpretation in $X$**.
Soundness and completeness

Theorem. Let $X$ be a set.

1. All tautologies (= derivable formulas) of the classical logic are valid with respect to interpretation in $X$. The classical logic is sound with respect to this interpretation.

2. If $X$ is non-empty, the tautologies (= derivable formulas) of the classical logic are the only formulas valid with respect to interpretation in $X$. The classical logic is complete with respect to this interpretation.

The language of classical logic does not distinguish different non-empty sets $X$. 
Soundness and completeness

**Theorem.** Let $X$ be a set.

1. All tautologies (=" derivable formulas) of the classical logic are valid with respect to interpretation in $X$. The classical logic is **sound** with respect to this interpretation.
Soundness and completeness

**Theorem.** Let $X$ be a set.

1. All tautologies ($\equiv$ derivable formulas) of the classical logic are valid with respect to interpretation in $X$. **The classical logic is sound** with respect to this interpretation.

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Soundness and completeness

**Theorem.** Let $X$ be a set.

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2. If $X$ is non-empty, the tautologies ($= \text{derivable formulas}$) of the classical logic are the only formulas valid with respect to interpretation in $X$. **The classical logic is complete** with respect to this interpretation.

The language of classical logic does not distinguish different non-empty sets $X$. 
Definition. A topological space is a set $X$ together with a collection of subsets of $X$, called open subsets, satisfying the following axioms:

- The empty subset and $X$ are open.
- The union of any collection of open subsets is also open.
- The intersection of any pair of open subsets is also open.
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- The empty subset and $X$ are open.
- The union of any collection of open subsets is also open.
- The intersection of any pair of open subsets is also open.

**Example.** $X = \mathbb{R}^n$. A subset $P$ of $X$ is open iff for any point $x$ in $P$, some open ball containing $x$ is contained in $P$. 

![Diagram](image-url)
Topological spaces

**Definition.** The complement of an open subset is called **closed**.
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**Definition.** Given a subset $P$ of $X$, the **interior** of $P$ is the largest open subset of $P$.

**Example.** $X = \mathbb{R}$, $P = [a, b]$, $\text{interior}(P) = (a, b)$. 
**Definition.** The complement of an open subset is called **closed**.

**Definition.** Given a subset $P$ of $X$, the **interior** of $P$ is the largest open subset of $P$.

**Example.** $X = \mathbb{R}$, $P = [a, b]$, $\text{interior}(P) = (a, b)$.

**Definition.** Let $X$ and $Y$ be topological spaces. Then $f: X \to Y$ is **continuous** if for any open subset $U$ of $Y$, $f^{-1}(U)$ is an open subset of $X$. 
Quantifiers

- “∀x” means “for all x”
- “∃x” means “there exists x”
Quantifiers

- “∀x” means “for all x”
- “∃x” means “there exists x”

**Example.** Let $P$ be a subset of $\mathbb{R}^2$. Then

$$\forall x \in P \ \exists r \in \mathbb{R} \left( (r > 0) \land \forall y \in \mathbb{R}^2 (\text{dist}(x, y) < r \rightarrow y \in P) \right)$$

means that $P$ is open.
Quantifiers

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**Example.** Let $P$ be a subset of $\mathbb{R}^2$. Then

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means that $P$ is open.

The language with quantifiers is very expressive but undecidable.
Compromise: modality

The classical logic is extended with an operator $\Box$. Interpretations of $\Box P$:

- $P$ is known
- $P$ is provable
- $P$ is computable
- $P$ is necessary
- $P$ will always be true
- $P$ will be true tomorrow
- $etc.$
S4: $\land$, $\lor$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\square$

- Axioms of classical logic
- $\square P \rightarrow P$
- $\square P \rightarrow \square \square P$
- $\square (P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$
S4: $\land, \lor, \neg, \rightarrow, \leftrightarrow, \Box$

- Axioms of classical logic
  - $\Box P \rightarrow P$
  - $\Box P \rightarrow \Box \Box P$
  - $\Box (P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

- Rules of inference
  - $P, P \rightarrow Q \vdash Q$
  - $P \vdash \Box P$
  
- Topological interpretation of $\Box$:

Theorem. Let $X$ be a topological space. Then S4 is sound with respect to interpretation in $X$. 
S4: $\wedge, \lor, \neg, \rightarrow, \leftrightarrow, \Box$

- Axioms of classical logic
  - $\Box P \rightarrow P$
  - $\Box P \rightarrow \Box \Box P$
  - $\Box (P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

- Rules of inference
  - $P, P \rightarrow Q \quad \frac{}{Q}$
  - $P \quad \Box \frac{}{P}$

- Topological interpretation of $\Box$:
  - $\Box P = \text{interior}(P)$
S4: $\land, \lor, \neg, \rightarrow, \leftrightarrow, \square$

- Axioms of classical logic
  - $\square P \rightarrow P$
  - $\square P \rightarrow \square \square P$
  - $\square (P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$

Rules of inference

- $P, P \rightarrow Q \quad \Rightarrow \quad Q$
- $P \quad \Rightarrow \quad \square P$

Topological interpretation of $\square$:
$\square P = \text{interior}(P)$

**Theorem.** Let $X$ be a topological space. Then S4 is sound with respect to interpretation in $X$. 
**Theorem.** S4 is complete with respect to all interpretations in all topological spaces $X$, i.e. for any formula $F$, the following statements are equivalent:

1. $F$ is derivable in S4
2. $F$ is valid in each interpretation (for each topological space $X$)
**Theorem.** S4 is complete with respect to all interpretations in all topological spaces $X$, i.e. for any formula $F$, the following statements are equivalent:

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3. $F$ is valid in each interpretation for each $\mathbb{R}^n$
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1. $F$ is derivable in S4
2. $F$ is valid in each interpretation (for each topological space $X$)
3. $F$ is valid in each interpretation for each $\mathbb{R}^n$
4. $F$ is valid in each interpretation for some $\mathbb{R}^n$

**Corollary.** The modal logic (with operations $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\Box$) does not distinguish $\mathbb{R}^n$'s for different $n$. 
**Theorem.** S4 is complete with respect to all interpretations in all topological spaces $X$, i.e. for any formula $F$, the following statements are equivalent:

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4. $F$ is valid in each interpretation for some $\mathbb{R}^n$

**Corollary.** The modal logic (with operations $\land$, $\lor$, $\neg$, $\rightarrow$, $\square$) does not distinguish $\mathbb{R}^n$’s for different $n$. 

Start with a subset $S$ of $\mathbb{R}$. 
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

\begin{align*}
S & , 
\text{compl}(S) & , 
\text{inter}(S) & , 
\text{inter}(\text{compl}(S)) & , 
\text{compl}(\text{inter}(S)) & , 
\text{compl}(\text{inter}(\text{compl}(S))) & , 
\text{inter}(\text{compl}(\text{inter}(S))) & , 
\text{compl}(\text{inter}(\text{compl}(\text{compl}(S)))) & , 
\text{inter}(\text{compl}(\text{inter}(\text{compl}(S)))) & , 
\text{compl}(\text{inter}(\text{compl}(\text{compl}(\text{compl}(S))))) & , 
\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{compl}(S)))))) & , 
\ldots
\end{align*}

Can there be infinitely many different sets in these sequences? If not, what is the maximum number of different sets?
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

$$S$$
Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

\[
S \\
\text{inter}(S)
\]
Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

- $S$
- $\text{inter}(S)$
- $\text{compl}(\text{inter}(S))$
Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

$S$

inter$(S)$

compl$(\text{inter}(S))$

inter$(\text{compl}(\text{inter}(S)))$

...
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

$S$
inter($S$)
compl(inter($S$))
inter(compl(inter($S$)))
:::

Can there be infinitely many different sets in these sequences? If not, what is the maximum number of different sets?
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

- $S$
- $\text{inter}(S)$
- $\text{compl}(\text{inter}(S))$
- $\text{compl}(\text{inter}(\text{compl}(\text{inter}(S))))$
- $\vdots$

Can there be infinitely many different sets in these sequences? If not, what is the maximum number of different sets?
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

- $S$
- $\text{inter}(S)$
- $\text{compl}(\text{inter}(S))$
- $\text{inter}(\text{compl}(\text{inter}(S)))$
- ...
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

- $S$
- $\text{inter}(S)$
- $\text{compl}(\text{inter}(S))$
- $\text{inter}(\text{compl}(\text{inter}(S)))$
- $\vdots$
- $\text{compl}(S)$
- $\text{inter}(\text{compl}(S))$
- $\text{compl}(\text{inter}(\text{compl}(S)))$

Can there be infinitely many different sets in these sequences? If not, what is the maximum number of different sets?
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

\begin{align*}
S \\
\text{inter}(S) \\
\text{compl}(\text{inter}(S)) \\
\text{inter}(\text{compl}(\text{inter}(S))) \\
\vdots
\end{align*}

\begin{align*}
\text{compl}(S) \\
\text{inter}(\text{compl}(S)) \\
\text{compl}(\text{inter}(\text{compl}(S))) \\
\text{inter}(\text{compl}(\text{inter}(\text{compl}(S)))) \\
\vdots
\end{align*}
Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

\[
\begin{align*}
S & \quad \text{compl}(S) \\
\text{inter}(S) & \quad \text{inter(}\text{compl}(S)\text{)} \\
\text{compl(}\text{inter}(S)\text{))} & \quad \text{compl(}\text{inter(}\text{compl}(S)\text{)})\text{)} \\
\text{inter(}\text{compl(}\text{inter}(S)\text{))}\text{)} & \quad \text{inter(}\text{compl(}\text{inter(}\text{compl}(S)\text{)}\text{)})\text{)} \\
\vdots & \quad \vdots
\end{align*}
\]

Can there be infinitely many different sets in these sequences? If not, what is the maximum number of different sets?
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

- $S$
- $\text{inter}(S)$
- $\text{compl}(\text{inter}(S))$
- $\text{inter}(\text{compl}(\text{inter}(S)))$
- $\vdots$
- $\text{compl}(S)$
- $\text{inter}(\text{compl}(S))$
- $\text{compl}(\text{inter}(\text{compl}(S)))$
- $\text{inter}(\text{compl}(\text{inter}(\text{compl}(S))))$
- $\vdots$

Can there be infinitely many different sets in these sequences?
Problem

Start with a subset $S$ of $\mathbb{R}$. Consider the following sequences:

$S$
inter($S$)
compl(inter($S$))
inter(compl(inter($S$)))

Can there be infinitely many different sets in these sequences?

If not, what is the maximum number of different sets?
Example 1
Example 1

\[ \downarrow \text{interior} \]
Example 1

\[
\begin{align*}
\text{interior} & \\
\text{complement} & \\
\end{align*}
\]
Example 1

\[
\begin{align*}
\text{interior} & \quad \downarrow \quad \text{complement} \\
\text{interior} & \quad \downarrow \\
\end{align*}
\]
Example 1

Get 4 different subsets of $\mathbb{R}$
Example 2

Get 6 different subsets of $\mathbb{R}$
Example 2

\[ \text{interior} \]

Get 6 different subsets of $\mathbb{R}$
Example 2

Get 6 different subsets of $\mathbb{R}$.
Example 2

\[
\begin{align*}
\text{interior} & \downarrow \\
\text{complement} & \downarrow \\
\text{interior} & \downarrow \\
\end{align*}
\]
Example 2

Get 6 different subsets of $\mathbb{R}$
Example 2

Get 6 different subsets of \( \mathbb{R} \)
Example 2

Get 6 different subsets of $\mathbb{R}$
Example 3

\[ \text{complement} \rightarrow \text{interior} \rightarrow \text{interior} \rightarrow \text{complement} \rightarrow \text{complement} \rightarrow \text{interior} \rightarrow \text{complement} \]

Get 8 different subsets of \( \mathbb{R} \).
Example 3

Get 8 different subsets of $R$.
Example 3

\[
\begin{array}{c}
\bullet \quad \bullet \\
\downarrow \quad \text{interior} \\
\circ \quad \circ \\
\downarrow \quad \text{complement} \\
\bullet \quad \bullet
\end{array}
\]

Get 8 different subsets of \( \mathbb{R} \).
Example 3

\[
\begin{align*}
\text{complement} & \quad \Downarrow \quad \text{interior} \\
\text{complement} & \quad \Downarrow \quad \text{complement} \\
\text{interior} & \quad \Downarrow \quad \text{interior}
\end{align*}
\]

Get 8 different subsets of \( \mathbb{R} \).
Example 3

Get 8 different subsets of $\mathbb{R}$
Example 3

Get 8 different subsets of $\mathbb{R}$
Example 3

\[ \begin{align*}
\text{complement} & \quad \downarrow \quad \text{interior} \\
\text{complement} & \quad \downarrow \quad \text{interior} \\
\text{complement} & \quad \downarrow \quad \text{interior} \\
\text{complement} & \quad \downarrow \quad \text{interior}
\end{align*} \]
Example 3

Get 8 different subsets of $\mathbb{R}$
Example 3

Get 8 different subsets of $\mathbb{R}$
Problem

Can there be infinitely many different sets?

Answer: No.

What is the largest possible number of different sets?

Answer: 14.

Proof that we cannot get more than 14.

Lemma. There are at most 7 different sets in the sequence

\[ S \cap (S \cap (S \cap ...)) \cap \ldots \]

because

\[ S \cap (S \cap (S \cap ...)) = S \cap (S \cap (S \cap ...)) \cap \ldots \]
Problem

Can there be infinitely many different sets?
Answer: No.
Can there be infinitely many different sets?
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What is the largest possible number of different sets?
Problem

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Proof that we cannot get more than 14.
Problem

Can there be infinitely many different sets?
Answer: No.

What is the largest possible number of different sets?
Answer: 14.

Proof that we cannot get more than 14.
Lemma. There are at most 7 different sets in the sequence
\[ S, \quad \text{inter}(S), \quad \text{compl}(\text{inter}(S)), \quad \text{inter}(\text{compl}(\text{inter}(S))), \ldots \]

because

\[ \text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(S))))))) = \text{inter}(\text{compl}(\text{inter}(S))). \]
Proof

**Lemma.** $\square \neg \neg \neg \neg S = \square \neg \neg S$
**Lemma.** $\Box \neg \Box \neg \Box S = \Box \neg \Box S$

**Proof.** Let $T = \neg S$, then $S = \neg T$. We want to prove: $\Box \neg \Box \neg \Box \neg T = \Box \neg \Box \neg T$. 
Lemma. \(\square \neg \square \neg \square S = \square \neg S\)

Proof. Let \(T = \neg S\), then \(S = \neg T\). We want to prove:
\(\square \neg \square \neg \square \neg \square T = \square \neg \square \neg \square T\).

Notation: \(\diamond R \equiv \neg \square \neg R\).
Lemma. \( \Box \neg \Box \neg \Box \neg \Box S = \Box \neg \Box S \)

Proof. Let \( T = \neg S \), then \( S = \neg T \). We want to prove:
\( \Box \neg \Box \neg \Box \neg \Box T = \Box \neg \Box \neg \Box T \).

Notation: \( \Diamond R \equiv \neg \Box \neg R \).

In the topological interpretation “\( \Diamond R \)” means “the closure of \( R \)”.
Proof

**Lemma.**  $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

**Proof.** Let $T = \neg S$, then $S = \neg T$. We want to prove: $\Box\neg\Box\neg\Box\neg\Box T = \Box\neg\Box\neg\Box T$.

**Notation:** $\Diamond R \equiv \neg\Box\neg R$.

In the topological interpretation “$\Diamond R$” means “the closure of $R$”. Want to prove: $\Box\Diamond\Box\Diamond T \equiv \Box\Diamond T$. 

Lemma. $\square\neg\square\neg\square\neg\square S = \square\neg\square S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\square\neg\square\neg\square\neg\square T = \square\neg\square\neg\square T$.

Notation: $\Diamond R \equiv \neg\square\neg\neg R$.
In the topological interpretation “$\Diamond R$” means “the closure of $R$”.

Want to prove: $\square\Diamond\square\square \equiv \square\Diamond \square$.

Proof of $\square\Diamond \square \rightarrow \square\Diamond\square\square$. Axiom: $\square P \rightarrow P$

Let $P = \neg R$, then $\square\neg R \rightarrow \neg R$

Contrapositive: $R \rightarrow \neg\square\neg R$

Let $R = \square Q$, then $\square Q \rightarrow \neg\square\neg\square Q$

i.e. $\square Q \rightarrow \Diamond\square Q$

Apply $\square$: $\square\square Q \rightarrow \square\Diamond\square Q$

Axiom: $\square Q \rightarrow \square\square Q$

Therefore $\square Q \rightarrow \Diamond\square\square Q$

Let $Q = \Diamond T$, then $\square\Diamond T \rightarrow \square\Diamond\square\square T$. 
Lemma. \( \square \neg \square \neg \square \neg \square S = \square \neg \square S \)

Proof. Let \( T = \neg S \), then \( S = \neg T \). We want to prove: \( \square \neg \square \neg \square \neg \neg T = \square \neg \square \neg T \).

Notation: \( \Diamond R \equiv \neg \square \neg R \).

In the topological interpretation “\( \Diamond R \)” means “the closure of \( R \)”.

Want to prove: \( \square \Diamond \square \square T \equiv \square \Diamond T \).

Proof of \( \square \Diamond T \rightarrow \square \Diamond \square \square T \). Axiom: \( \square P \rightarrow P \)

Let \( P = \neg R \), then \( \square \neg R \rightarrow \neg R \)

Contrapositive: \( R \rightarrow \neg \square \neg R \)

Let \( R = \square Q \), then \( \square Q \rightarrow \neg \square \neg \square Q \)

i.e. \( \square Q \rightarrow \Diamond \square Q \)

Apply \( \square \): \( \square \square Q \rightarrow \square \Diamond \square Q \)

Axiom: \( \square Q \rightarrow \square \square Q \)

Therefore \( \square Q \rightarrow \square \Diamond \square Q \)

Let \( Q = \Diamond T \), then \( \square \Diamond T \rightarrow \square \Diamond \square \square T \).

Similarly \( \square \Diamond \square \square T \rightarrow \square \Diamond T \).
Similarly, there are at most 7 different subsets in the sequence
\[
\text{compl}(S) \\
\text{inter}(\text{compl}(S)) \\
\text{compl}(\text{inter}(\text{compl}(S))) \\
\text{inter}(\text{compl}(\text{inter}(\text{compl}(S)))) \\
\vdots
\]
because
\[
\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{inter}(\text{compl}(S)))))),) = \\
\text{inter}(\text{compl}(\text{inter}(\text{compl}(\text{S})))
\]
so at most 14 different subsets total.
Proof

Similarly, there are at most 7 different subsets in the sequence
\[
\text{compl}(S) \\
\text{inter}\left(\text{compl}(S)\right) \\
\text{compl}\left(\text{inter}(\text{compl}(S))\right) \\
\text{inter}\left(\text{compl}\left(\text{inter}\left(\text{compl}(S)\right)\right)\right) \\
\vdots \\
because
\text{inter}\left(\text{compl}\left(\text{inter}\left(\text{compl}\left(\text{inter}\left(\text{compl}(S)\right)\right)\right)\right)\right) = \\
\text{inter}\left(\text{compl}\left(\text{inter}\left(\text{compl}(S)\right)\right)\right),
\]
so at most 14 different subsets total.

**Homework problem.** Find a subset of $\mathbb{R}$ for which you get 14 different subsets.
Definition. A dynamic topological system is a topological space $X$ with a continuous function $f: X \to X$. 
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New modal operator $\circ : \circ P$ is interpreted as $f^{-1}(P)$. 
**Dynamic topological systems**

**Definition.** A dynamic topological system is a topological space $X$ with a continuous function $f: X \rightarrow X$. New modal operator $\circ$: $\circ P$ is interpreted as $f^{-1}(P)$.

**S4C**

- Axioms of classical logic
  - $\Box P \rightarrow P$
  - $\Box P \rightarrow \Box \Box P$
  - $\Box (P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
  - $\circ (P \rightarrow Q) \rightarrow (\circ P \rightarrow \circ Q)$
  - $(\circ \neg P) \leftrightarrow (\neg \circ P)$
  - $(\circ \Box P) \leftrightarrow (\Box \circ \Box P)$

- Rules of inference
  1. $P, P \rightarrow Q \quad \frac{}{Q}$
  2. $\Box P \quad P$  
  3. $\circ P \quad P$
**Theorem.** Let $F$ be a formula. The following are equivalent:

1. $F$ is derivable in S4C
2. $F$ is valid with respect to every interpretation in every topological space
3. $F$ is valid with respect to every interpretation in every $\mathbb{R}^n$

However, the above statements are not equivalent to

4. $F$ is valid with respect to every interpretation in $\mathbb{R}$

Namely, there exists a formula that is valid in $\mathbb{R}$ but not valid in any $\mathbb{R}^n$ with $n > 1$.

**Corollary.** The language of S4C distinguishes $\mathbb{R}$ from $\mathbb{R}^n$ for $n > 1$. 
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However, the above statements are not equivalent to

4. $F$ is valid with respect to every interpretation in $\mathbb{R}$

Namely, there exists a formula that is valid in $\mathbb{R}$ but not valid in any $\mathbb{R}^n$ with $n > 1$.

**Corollary.** The language of S4C distinguishes $\mathbb{R}$ from $\mathbb{R}^n$ for $n > 1$. 
Example

Let \( U = \Box P \) (\( U \) is open),
\( \Phi = (\Diamond U) \land (\Diamond \neg U) \) (\( \Phi \) is the boundary of \( U \)),

Lemma.
If \( P \) and \( Q \) are subsets of \( \mathbb{R} \), then \( \Psi = \emptyset \).

Corollary.
\( \neg \Psi = \mathbb{R} \).

Lemma.
There exist subsets \( P \) and \( Q \) of \( \mathbb{R} \) and a continuous function \( f: \mathbb{R} \to \mathbb{R} \) such that \( \Psi \neq \emptyset \), i.e. \( \neg \Psi \neq \mathbb{R} \).

Corollary.
The formula \( \neg \Psi \) is not derivable in S4C.
Example

Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \land (\Diamond \neg U)$ (Φ is the boundary of U),
$\Psi = (\Box \lozenge \Phi) \land (\lozenge Q) \land (\Diamond \lozenge \neg Q)$. 

Lemma.
If $P$ and $Q$ are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

Corollary.
$\neg \Psi = \mathbb{R}$.

Lemma.
There exist subsets $P$ and $Q$ of $\mathbb{R}$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi \neq \emptyset$, i.e. $\neg \Psi \neq \mathbb{R}^2$.

Corollary.
The formula $\neg \Psi$ is not derivable in S4C.
Example

Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \land (\Diamond \neg U)$ (\Phi is the boundary of \textit{U}),
$\Psi = (\Box \Box \Phi) \land (\Box Q) \land (\Diamond \Box \neg Q)$.

**Lemma.** If \textit{P} and \textit{Q} are subsets of \(\mathbb{R}\), then \(\Psi = \emptyset\).
Let \( U = \Box P \) (\( U \) is open),
\[ \Phi = (\Diamond U) \land (\Diamond \neg U) \] (\( \Phi \) is the boundary of \( U \)),
\[ \Psi = (\Box \diamond \Phi) \land (\diamond \Box Q) \land (\Diamond \Diamond \neg Q). \]

**Lemma.** If \( P \) and \( Q \) are subsets of \( \mathbb{R} \), then \( \Psi = \emptyset \).

**Corollary.** \( \neg \Psi = \mathbb{R} \)
Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \land (\Diamond \neg U)$ (\Phi is the boundary of U),
$\Psi = (\Box \circ \Phi) \land (\circ Q) \land (\Diamond \circ \neg Q)$.

**Lemma.** If $P$ and $Q$ are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

**Corollary.** $\neg \Psi = \mathbb{R}$

**Lemma.** There exist subsets $P$ and $Q$ of $\mathbb{R}^2$ and a continuous function $f: \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi \neq \emptyset$, i.e. $\neg \Psi \neq \mathbb{R}^2$. 
Example

Let $U = \Box P$ ( $U$ is open),
$\Phi = (\Diamond U) \land (\Diamond \neg U)$ ( $\Phi$ is the boundary of $U$),
$\Psi = (\Box \Diamond \Phi) \land (\Diamond \Box Q) \land (\Diamond \Diamond \neg Q)$.

**Lemma.** If $P$ and $Q$ are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

**Corollary.** $\neg \Psi = \mathbb{R}$

**Lemma.** There exist subsets $P$ and $Q$ of $\mathbb{R}^2$ and a continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\Psi \neq \emptyset$, i.e. $\neg \Psi \neq \mathbb{R}^2$.

**Corollary.** The formula $\neg \Psi$ is not derivable in $S4C$. 
Theorem

(joint work with A. Nogin; also by D.F. Duque)

For any $n \geq 2$, S4C is complete with respect to any interpretation in $\mathbb{R}^n$. 
(joint work with A. Nogin)

The following formulas are valid with respect to any interpretation in $\mathbb{R}$:

$$\lozenge Q \land \Diamond (\lozenge \neg Q \land \lozenge \neg P \land \Box \lozenge P) \rightarrow \Diamond (\lozenge \neg Q \land \lozenge \neg P \land \Box \lozenge P)$$

$$\lozenge \neg P \land \lozenge \neg Q \land \Diamond \Box \lozenge P \land \Diamond \lozenge \neg P \land \Diamond Q \land \Box \lozenge S \rightarrow \Diamond (\Box \lozenge P \land \Diamond \neg P \land \Box \Box S)$$
(joint work with A. Nogin)

The following formulas are valid with respect to any interpretation in $\mathbb{R}$:

$\Box Q \land \Diamond (\Diamond \neg Q \land \Box \Diamond \neg P \land \Box \Box P) \rightarrow \Diamond (\Diamond \neg Q \land \Diamond \neg P \land \Diamond \Box \Box P)$

$\Box \neg P \land \Box \neg Q \land \Box \Diamond \Box P \land \Diamond \Diamond (\neg P \land Q) \land \Box \Box S \rightarrow$

$\Diamond (\Diamond \Box P \land \Diamond \Box \neg P \land \Box \Box S)$

Open question

What exactly is the dynamic topological logic of $\mathbb{R}$?
Application: Hybrid Control Systems

- “Discrete” parameters: Discrete Mathematics
- “Continuous” parameters: Optimal Control Theory: Differential Equations, PDEs, etc
- Parameters of both types: Hybrid Control System: Modal Logic
Application: Hybrid Control Systems

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Thank you!