Autotopism groups in translation planes

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Finite translation planes

Let $\mathbb{F}$ be a field of order $q = p^h$, where p is prime, and let V be an $2n$-dimensional vector space over $\mathbb{F}$.

- A spread $\mathcal{S}$ of V is a set of $q^n + 1$ n-dimensional subspaces of V with trivial pairwise intersection.
- The incidence geometry with points and lines as described below is called a (finite) translation plane of order q^n:
  1. The elements/vectors in V are called points,
  2. The subspaces in $\mathcal{S}$ and their translates are called lines.
  3. The incidence is the obvious set-theoretic one.
- There is another definition of translation planes, but every translation plane π can be obtained using the construction above (André or Bruck-Bose).

Affine/Projective planes

Let π be a translation plane.

- Every line on π contains q^n points.
- Every point of π is on $q^n + 1$ lines.
- π contains q^{2n} points and $q^{2n} + q^n$ lines.

Oscar, draw a picture!

By adding a line (called ℓ_∞) to π we get a different type of plane Π (called the projective extension of π).

- Every line on Π contains $q^n + 1$ points.
- Every point of Π is on $q^n + 1$ lines.
- Π contains $q^{2n} + q^n + 1$ points and $q^{2n} + q^n + 1$ lines.

Collineations

Let π be a translation plane of order q^n with associated spread $\mathcal{S}$ of a vector space V.

- A bijective function ϕ on the points of π that preserves incidence is called a collineation of π.
- It is known (fundamental theorem of projective geometry) that if ϕ fixes the origin, then it is a semi-linear function $\phi : V \to V$.
- Note that a collineation of a translation plane π extends in a natural way to a collineation of its projective extension Π. Moreover, such a collineation fixes ℓ_∞.
- If a collineation of π fixes a line (of π) then it fixes a point on ℓ_∞.

Perspectivities: Elations and homologies

Let ϕ be a collineation of a translation plane π.

- If ϕ fixes a line ℓ pointwise, then it also fixes a point C, and every line through C (setwise).
- If ϕ fixes a point C, and every line through C (setwise), then it also fixes a line ℓ pointwise.
- If ϕ fixes a line ℓ and a point C as above then ϕ is said to be a perspectivity:
    1. If $C \in \ell$ then ϕ is an elation.
    2. If $C \notin \ell$ then ϕ is a homology.

Homologies and autotopisms

Let ϕ be a collineation of a translation plane π.

- If ϕ fixes a point $C \neq 0$ then the line $0C$ is fixed by ϕ.
- If ϕ is a homology then ϕ fixes two non-parallel lines, and thus it fixes two points on ℓ_∞.
- Homologies are very important, as they have been used often to obtain group-theoretical characterizations of translation planes.

Definition

A collineation ϕ of a translation plane π is said to be an autotopism of π if it fixes at least two points on ℓ_∞.

The problem

- Let π be a translation plane of order q^n. Assume π admits a group of collineations G of order $q^n - 1$.

  What can we say about π?

- The obvious example of such a situation is a plane admitting a cyclic autotopism group of order $q^n - 1$.

Theorem (Jha & Johnson)

Let π be a translation plane of order $q^n \neq 2^6$ that admits an autotopism group which is transitive on the remaining (infinite) points. Then, the full collineation group fixes the autotopism triangle, or [except for a few cases] the plane is Desarguesian.

What do we need?

Let π be a translation plane of order q^n. Assume π admits a group of collineations G of order $q^n - 1$. What can we say about π?

This problem seems too hard. Do we need to ask for extra hypotheses?

We ask for

- G is a linear group,
- G must act faithfully on ℓ_∞, and
- **G must be cyclic.**

What do we need?

Let π be a translation plane of order q^n. Assume π admits a group of collineations G of order $q^n - 1$. What can we say about π?

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We ask for

- G is a linear group,
- G must act G acts faithfully on ℓ_∞, and
- **G must be Abelian.**

What do we need?

Let π be a translation plane of order q^n. Assume π admits a group of collineations G of order $q^n - 1$. What can we say about π?

This problem seems too hard. Do we need to ask for extra hypotheses?

We ask for

- G is a linear group,
- G must act G acts faithfully on ℓ_∞, and
- **G must be solvable (or nilpotent?).**

Results I

When G is assumed to be Cyclic.

- G is an autotopism group, or
- G is a Baer autotopism group.
- In either case. There are two symmetric homology groups of order $q-1$, and thus the plane can be associated to a (possibly partial) flat flock of a Segre variety.
- In certain cases we are able to prove that the plane must be a $jj\cdots j$-plane.

Results II

When G is assumed to be solvable/nilpotent this is still work in progress.

We think we will get:

- If the order of π is odd (hence the assumption of solvable is unnecessary) then G (or $Fit(G)$) is an autotopism group, or
- If the order of π is even then G (or $Fit(G)$) is an autotopism group or a Baer autotopism group.
- We do not know yet about having two symmetric homology groups of order $q-1$. If we got that then we would be able to associate the plane to a (possibly partial) flat flock of a Segre variety
- We think we might need to assume many extra hypotheses to get that these planes are $jj\cdots j$-planes. We suspect they will be graded $jj\cdots j$-planes.

Thank you!

Any questions?