Autotopism groups and $jj\cdots j$-planes

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Finite Translation Planes

Let $\mathbb{F}$ be a field of order $q = p^h$, where $p$ is prime, and let $V$ be a $2n$-dimensional vector space over $\mathbb{F}$.

- A spread $S$ of $V$ is a set of $q^n + 1$ $n$-dimensional subspaces of $V$ with trivial pairwise intersection.

- (André / Bruck-Bose) A finite translation plane of order $q^n$ is the incidence geometry with points and lines as described below:
  1. The elements/vectors in $V$ are called points,
  2. The subspaces in $S$ and their translates are called lines.
  3. The incidence is the obvious set-theoretic one.

- By adding a line (called $\ell_{\infty}$) to $\pi$ consisting of all the ‘slopes’ of lines of $\pi$ we get a different type of plane $\Pi$ (the projective extension of $\pi$).
Collineations
Let $\pi$ be a translation plane of order $q^n$ with associated spread $S$ of a vector space $V$.

- A bijective function $\phi$ on the points of $\pi$ that preserves incidence is called a collineation of $\pi$.

- Note that a collineation of a translation plane $\pi$ extends in a natural way to a collineation of its projective extension $\Pi$.

- If $\phi$ fixes a line $l$ pointwise, then it also fixes a point $C$, and every line through $C$ (setwise). The converse of this is also true.

- If $\phi$ fixes a line $l$ and a point $C$ as above then $\phi$ is said to be a perspectivity:
  1. If $C \in l$ then $\phi$ is an elation.
  2. If $C \notin l$ then $\phi$ is a homology.

- Every linear (affine) homology fixes two points on $l_\infty$. 
Definition

A collineation $\phi$ of a translation plane $\pi$ is said to be an autotopism of $\pi$ if it fixes at least two points on $\ell_{\infty}$.

Problem: Let $\pi$ be a translation plane of order $q^n$. Assume $\pi$ admits a linear group of collineations $G$ of order $q^n - 1$.

When can we assure that $\pi$ admits an autotopism group?

This problem, as it is, is too hard. We ask for the following extra hypotheses:

- $G$ must act faithfully on $\ell_{\infty}$, and
- $G$ must be cyclic / Abelian / solvable / nilpotent.
**jj · · · j-planes: Problem II**

**Definition**

Let $F \subset M_n(q)$ be a field of order $q^n$, and let $j_2, j_3, \ldots, j_n$ be elements of $\{0, 1, 2, \ldots, q - 2\}$. A $jj \cdots j$-plane is a translation plane with spread $S$ in $V_{2n}$ given by the orbit of the subspace $y = x$ under the group

$$G = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} ; M \in F \right\}$$

union the subspace $x = 0$, where

$$\Delta_M = diag(1, \partial^{j_2}, \ldots, \partial^{j_n})$$

for all $M \in M_n(q)$, where $\partial = \det(M)$.

**Problem:** Find a geometric characterization of these planes.
Assume $G$ is Cyclic (most results generalize for Abelian). Then,

- $G$ is an autotopism group, or
- $G$ is a Baer autotopism group.

In either case, there are two symmetric homology groups of order $q - 1$, and thus the plane can be associated to a (possibly partial) flat flock of a Segre variety.

In certain cases we are able to prove that the plane must be a $jj \cdots j$-plane.
Theorem

When $G$ is assumed to be solvable (nilpotent). Then,

- If the order of $\pi$ is odd (hence the assumption of solvable is unnecessary) then $G$ (or $\text{Fit}(G)$) is an autotopism group, or
- If the order of $\pi$ is even then $G$ (or $\text{Fit}(G)$) is an autotopism group or a Baer autotopism group.
- We do not know yet about having two symmetric homology groups of order $q - 1$. If we got that then we would be able to associate the plane to a (possibly partial) flat flock of a Segre variety
- We think we might need to assume many extra hypotheses to get that these planes are $jj \cdots j$-planes. We suspect they will be graded $jj \cdots j$-planes.
Thank you!

Any questions?