

Autotopism groups and $jj \cdots j$ -planes

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Finite Translation Planes

Let \mathbb{F} be a field of order $q = p^h$, where p is prime, and let V be a $2n$ -dimensional vector space over \mathbb{F} .

- A spread \mathcal{S} of V is a set of $q^n + 1$ n -dimensional subspaces of V with trivial pairwise intersection.
- (André / Bruck-Bose) A finite translation plane of order q^n is the incidence geometry with points and lines as described below:
 - 1 The elements/vectors in V are called points,
 - 2 The subspaces in \mathcal{S} and their translates are called lines.
 - 3 The incidence is the obvious set-theoretic one.
- By adding a line (called ℓ_∞) to π consisting of all the 'slopes' of lines of π we get a different type of plane Π (the projective extension of π).

Collineations

Let π be a translation plane of order q^n with associated spread \mathcal{S} of a vector space V .

- A bijective function ϕ on the points of π that preserves incidence is called a collineation of π .
- Note that a collineation of a translation plane π extends in a natural way to a collineation of its projective extension Π .
- If ϕ fixes a line ℓ pointwise, then it also fixes a point C , and every line through C (setwise). The converse of this is also true.
- If ϕ fixes a line ℓ and a point C as above then ϕ is said to be a perspectivity:
 - 1 If $C \in \ell$ then ϕ is an elation.
 - 2 If $C \notin \ell$ then ϕ is a homology.
- Every linear (affine) homology fixes two points on ℓ_∞ .

Autotopisms: Problem I

Definition

A collineation ϕ of a translation plane π is said to be an autotopism of π if it fixes at least two points on ℓ_∞ .

Problem: Let π be a translation plane of order q^n . Assume π admits a linear group of collineations G of order $q^n - 1$.

When can we assure that π admits an autotopism group?

This problem, as it is, is too hard. We ask for the following extra hypotheses:

- G must act faithfully on ℓ_∞ , and
- G must be cyclic / Abelian / solvable / nilpotent.

$jj \cdots j$ -planes: Problem II

Definition

Let $F \subset M_n(q)$ be a field of order q^n , and let j_2, j_3, \dots, j_n be elements of $\{0, 1, 2, \dots, q-2\}$. A $jj \cdots j$ -plane is a translation plane with spread S in V_{2n} given by the orbit of the subspace $y = x$ under the group

$$G = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix}; M \in F \right\}$$

union the subspace $x = 0$, where

$$\Delta_M = \text{diag}(1, \partial^{j_2}, \dots, \partial^{j_n})$$

for all $M \in M_n(q)$, where $\partial = \det(M)$.

Problem: Find a geometric characterization of these planes.

Results I

Theorem

Assume G is Cyclic (most results generalize for Abelian). Then,

- G is an autotopism group, or
- G is a Baer autotopism group.
- In either case. There are two symmetric homology groups of order $q - 1$, and thus the plane can be associated to a (possibly partial) flat flock of a Segre variety.
- In certain cases we are able to prove that the plane must be a $jj \cdots j$ -plane.

Results II

Theorem

When G is assumed to be solvable (nilpotent). Then,

- *If the order of π is odd (hence the assumption of solvable is unnecessary) then G (or $\text{Fit}(G)$) is an autotopism group, or*
- *If the order of π is even then G (or $\text{Fit}(G)$) is an autotopism group or a Baer autotopism group.*
- *We do not know yet about having two symmetric homology groups of order $q - 1$. If we got that then we would be able to associate the plane to a (possibly partial) flat flock of a Segre variety*
- *We think we might need to assume many extra hypotheses to get that these planes are $jj \cdots j$ -planes. We suspect they will be graded $jj \cdots j$ -planes.*

Thank you!

Any questions?