

Oscar Vega. Department of Mathematics.  
College of Science and Mathematics Centennial Celebration.  
California State University, Fresno. May 13<sup>th</sup>, 2011.

## The Game

A Sudoku board is a  $9 \times 9$  table filled with the numbers  $1, 2, \dots, 9$ , that has been partitioned into nine  $3 \times 3$  blocks such that:

- each symbol appears exactly once in each row,
- each symbol appears exactly once in each column,
- each symbol appears exactly once in each block.

The game consists in completing a sudoku board from the few entries given.

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

# Sudo What?

- The name is derived from 'suji wa dokushin ni kagiru', which is the Japanese for 'the digits must remain single'.
- It is known that this game (essentially the same) was already known in France in the 1800's.

**Thompson-Elysee.** — M. Léon Faivre, âgé de 36 ans, demeurant 24, rue de Valenciennes, s'occupait, en l'honneur de l'avenue des Ambassadeurs, un flacon et le portait sous le costume, mais à terre, en proie à l'émotion, on le porta à l'hôpital Beaujon, où on le soigna avec de la morphine et de la digitale.

est très grave, n'a pu se lever, et a dû être transféré à l'hôpital Beaujon, où on le soigna avec de la morphine et de la digitale.

**Il se venge.** — Deux des grands magasins de Paris ont été, au cours de la semaine dernière, l'objet de faits de violence, un d'eux, nommé l'Éclair, a été incendié.

**Un vagabond.** — Un vagabond, nommé l'Éclair, a été arrêté et conduit au dépôt.

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ont la tête prise et tressée par les tampons. La mort a été instantanée.

LEOPOLD LARNA.

## DIVERTISSEMENTS QUOTIDIENS

N° 2879 — CARÉ MAGIQUE DIABOLIQUE  
Par M. E. Meynel

Compléter le caré ci-dessous en employant les neuf premiers nombres chacun neuf fois de manière que les horizontales, les verticales et les deux grandes diagonales donnent toujours à l'addition le même total.

7	8	9	1	2	3	4	5	6
3			4					8
5			9					1
6			3					4
1	8	3	1	5	6	7	8	9
8				7				8
9				1				5
7				8				7
1	5	6	7	8	9	1	2	3

Ce caré devra être diabolique, c'est-à-dire que le carré contenu dans n'importe quelle ligne horizontale ou une colonne verticale à la suite de toutes les autres.

## N° 2883 — MATHÉMATIQUES

Par M. Andréon H.  
Solutions

Le marchand a vendu 27000 francs; le 30<sup>e</sup> jour, le dernier, il a vendu 500 francs.

Solutions justes

MM. Améthyste; un chercheur; Paul et Jules Duplant; Albert Lalari; L. Grollet; C. Gerbaud.

Les solutions et les essais de problèmes posés doivent être adressés, dans la huitaine, au rédacteur soussigné.

ÉLITE ANNÉE.

passer, ce sont, etc.

M. Auguste Germain sera représenté dans prochain.

Cet ouvrage complet sera série par M.

On vient de supprimer de nos programmes d'enseignement, et on a supprimé le latin.

Mlle Mathilde-Adolphe, élève du lycée de Valenciennes, a été admise à l'École normale supérieure de Saint-Denis.

Devant le très grand succès de l'École normale supérieure de Saint-Denis.

Immense succès, le succès de l'École normale supérieure de Saint-Denis.

PROGRAMME

OPERA, 8 h. 00. — Le Trouvère. — BELLE.

FRANÇAIS, 8 h. 1/2. —

OPERA-COMIQUE. —

ODEON. — Cléopâtre.

RENAISSANCE. —

GYMNASE. — Cléopâtre.

VAUDEVILLE. — Cléopâtre.

VARIÉTÉS. — Cléopâtre.

# Sudo-History

- The modern version of Sudoku is an American invention (not Japanese). The author is unknown, but there is evidence it was Howard Garns.
- First published by Dell Puzzle Magazine in 1979 under the name *Number Place*.
- Taken by the Japanese puzzle company Nikoli in the mid-80's. That is when the name Sudoku was originated.

# Pop Culture: Cool, Awesome... X-treme!!!

4	9	8		2			1
	5			1		2	
		6					9
8			4		5	6	
9			5				3
	3	4		1			7
7				2		5	8
	4		7		1		7
2		6	5	8	7	4	
						9	3
						7	6
						8	3
						2	4
						9	6
						5	
						8	7
						7	6

9			1	8	6			5
		4				6		
		5					9	
4			9	6	7			1
3			5		8			7
7			4	1	3			9
	4					2		
	7				5		4	
5			7	3	2			4

7			9	5	6			8
		9					4	5
		3						5
			1		5	6	9	
			5		1	3		2
			3		4	8	2	

				4		1		5
				3		4		7
				9		5		2

5			6	2	3			9
	8				1		9	
	3				6		5	4

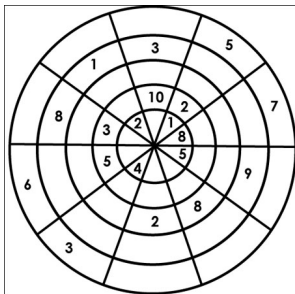
5						7	6	3
						4		6

8			3	6	5			1
9			8	7				6
7			9	1	2			4

7						2	8	6
						4	4	3
						4	7	5

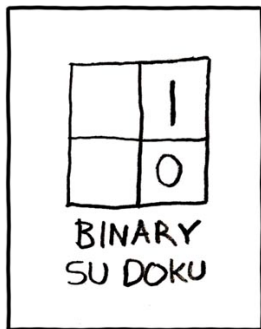
	2					5		
	7					2		
3		2	8	1				7

						1			9
						4			5
						9	1	7	8



		7	8						
						1			
				4	8			5	
				3	1				
			3			9			
			4	8					
2		6	1						
		5							
						7	6		

# Pop Culture: Funny,... WTH?



## Math Game?

People say that sudoku is a math game because numbers are used to play.

But any set of nine distinct symbols can be used to play.

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9



E	C			G				
F			A	I	E	C		
	I	H					F	
H				F				C
D			H		C			A
G				D				F
	F					B	H	
			D	A	I			E
				H			G	I

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4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9



◇	ℕ			♠				
♣			∫	∞	◇	ℕ		
	∞	⊗					♣	
H				♣				ℕ
∂			⊗		ℕ			∫
♠				∂				♣
	♣					♥	⊗	
			∂	∫	∞			◇
				⊗			♠	∞

Having to use numbers to fill the cells does not make it a math game.  
Then, what does?

# The Math of the Game

- In order to succeed solving Sudoku puzzles one has to use logic!
- In fact, many of the techniques used to play sudoku can be matched to techniques used by mathematicians to prove theorems.
- But that is another talk....
  
- In this talk I will only introduce a few areas of mathematics that are related to Sudoku. This list is not, by far, complete.

# Algebraic Combinatorics.

- How many Sudoku boards are there?
- More than the number of stars in the Universe.
- But, how many are *really* distinct?

5	3			7			
6			1	9	5	3	
	9	8					6
8				6			3
4			8		3		1
7				2			6
	6					2	8
			4	1	9		5
			8			7	9



5	6		8	4	7		
3		9				6	
		8					
	1			8			4
7	9		6		2		1
	5			3			9
	3					2	
		6				8	7
			3	1	6		5

## Algebraic Combinatorics.

We can create many Sudoku boards out of a given one by:

5	6		8	4	7			
3		9				6		
		8						
	1			8			4	
7	9		6		2		1	8
	5			3				9
	3					2		
		6				8		7
			3	1	6		5	9

- (i) transposing it,
- (ii) interchanging 'stacks',
- (iii) interchanging 'bands',
- (iv) interchanging columns in a stack,
- (v) interchanging rows in a band,
- (vi) relabeling of the digits.
- (vii) rotating it.

All these 'moves' create *group actions* on the set of all Sudoku boards.

## Algebraic Combinatorics.

Boards that can be connected to each other using (finitely many of) the tricks described in the previous slide are said to be *essentially the same*. How many boards can we get out of a single given board?

- Starting with one Sudoku puzzle, a daily calendar of (essentially the same) Sudoku puzzles can be produced... enough for the entire next century! <sup>1</sup>.
- There are 6,670,903,752,021,072,936,960 distinct Sudoku boards <sup>2</sup>.
- There are 5,472,730,538 *essentially different* boards <sup>3</sup>.
- All these numbers can be connected to each other by using Burnside's formula.

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<sup>1</sup>Bartlett & Langville. An Integer Programming Model for the Sudoku Model. The Journal of Online Mathematics and Its Applications, Vol. 8 (2008)

<sup>2</sup>Felgenhauer & Jarvis, Mathematics of Sudoku I (2006)

<sup>3</sup>Russel & Jarvis. Mathematics of Sudoku II (2006)

# Computational Mathematics

Q. What is the minimum number of 'givens' in a Sudoku board that are needed to have unique solution?

A. Unknown.

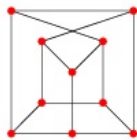
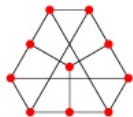
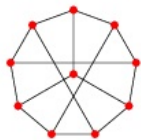
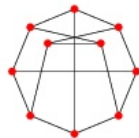
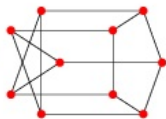
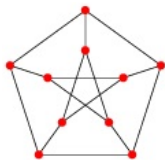
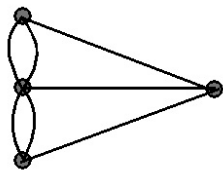
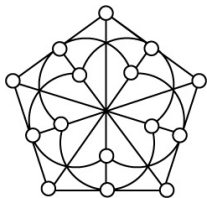
The minimum number known is 17. Gordon Royle has a collection of 49151 distinct Sudoku configurations with 17 entries at

<http://mapleta.maths.uwa.edu.au/~gordon/sudokumin.php>

- Gary McGuire is conducting a search for a 16-given Sudoku board with unique solution. He developed a program, called CHECKER for this purpose. It seems that the program requires about 300,000 years (on a one-core computer equipped with CPU, Intel(R) Xeon(R) E5520 @ 2.27GHz) to finish the job.

<http://www.math.ie/checker.html>

# Graph Theory



# Graph Theory

Herzberg and Ram Murty associated a graph  $G_S$  to any given Sudoku board  $S$ .

- vertices = cells.
- edge between two vertices = two cells that are 'buddies'.

				B				
				B				
				B				
B	B	B	B	C	B	B	B	B
			B	B	B			
			B	B	B			
				B				
				B				
				B				

# Graph Theory

Herzberg and Ram Murty<sup>4</sup> connected the idea of a unique solution for a Sudoku board to the concept of coloring in graphs.

## Theorem (Herzberg & Ram Murty)

*Let  $G$  be a graph with chromatic number  $C(G)$  and  $P$  be a partial coloring of  $G$  using only  $C(G) - 2$  colors. If  $P$  can be completed to a total proper coloring of  $G$ , then there are at least two ways of extending the coloring.*

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<sup>4</sup>Sudoku Squares and Chromatic Polynomials. Notices of the AMS. Vol. 54, Number 6 (2007)

## Latin Squares

- A Latin square of order  $n$  is an  $n \times n$  table filled with  $n$  different symbols in such a way that:  
each symbol appears exactly once in each row, and each column.
- A Latin square of order 9 with entries  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  :

8	3	2	5	9	1	6	7	4
1	8	5	7	4	6	3	9	2
5	7	1	2	6	4	9	8	3
4	9	6	3	8	7	2	5	1
2	6	7	9	5	3	4	1	8
9	4	3	8	1	2	7	6	5
7	1	4	6	3	8	5	2	9
3	2	9	1	7	5	8	4	6
6	5	8	4	2	9	1	3	7

**Relevant article:** R. Pedersen & T. Vis. Sets of Mutually Orthogonal Sudoku Latin Squares. The College Mathematics Journal. Vol 40, No 3. 2009.

## Gerechte Designs

- Gerechte designs were introduced by W.U. Behrens in 1956.
- A gerechte design is an  $n \times n$  table filled with  $n$  different symbols and partitioned into  $n$  regions, each containing  $n$  small squares of the grid, in such a way that :

each symbol occurs exactly once in each row, each column, and each region.

- Example: two gerechte designs of order 9:

6	9	5	3	4	2	8	1	7
8	7	9	6	3	1	5	4	2
1	5	2	8	6	9	4	7	3
3	1	4	5	2	7	6	9	8
5	3	7	4	9	6	2	8	1
9	4	6	1	8	3	7	2	5
7	2	8	9	5	4	1	3	6
4	8	1	2	7	5	3	6	9
2	6	3	7	1	8	9	5	4

8	3	5	4	1	6	9	2	7
2	9	6	8	5	7	4	3	1
4	1	7	2	9	3	6	5	8
5	6	9	1	3	4	7	8	2
1	2	3	6	7	8	5	4	9
7	4	8	5	2	9	1	6	3
6	5	2	7	8	1	3	9	4
9	8	1	3	4	5	2	7	6
3	7	4	9	6	2	8	1	5

## Other Areas

- R. Bailey, P. Cameron, and R. Connelly, Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes, Amer. Math. Monthly, 115 (2008), 383-403.
- E. Arnold, S. Lucas and L. Taalman. Gröbner Basis Representations of Sudoku The College Mathematics Journal. Vol. 41, No. 2. 2010.
- J. Gago-Vargas, I. Hartillo-Hermoso, J. Martín-Morales, and J.M. Ucha-Enriquez, Sudokus and Gröbner bases: not only a divertimento. Computer Algebra in Scientific Computing, 155 - 165, Lecture Notes in Comput. Sci., 4194, Springer, Berlin, 2006.
- A. Bartlett & A. Langville. An Integer Programming Model for the Sudoku Model. The Journal of Online Mathematics and Its Applications, Vol. 8 (2008)
- R. Fontana and M. Rogantin. Indicator function and sudoku designs. Algebraic and Geometric Methods in Statistics, pp. 203-224. Cambridge University Press (2010).

