

COUNT on me, I'll be DISCRETE

GAUSS, October 5, 2005

There are several ways to prove a fact, for example:

1. Proof by intimidation:

Trivial.

2. Proof by reference to inaccessible literature:

...Thus, because of the corollary of theorem 1 in the privately circulated memoir of the Slovenian Philological Society, 1883...

3. Proof by cosmology:

The negation of the proposition is unimaginable or meaningless.

4. Proof by ghost reference:

Nothing even remotely resembling the cited theorem appears in the reference given.

...and there are also beautiful simple proofs like the one showing the existence of infinitely many prime numbers or the one that shows $0,99999\dots = 1$.

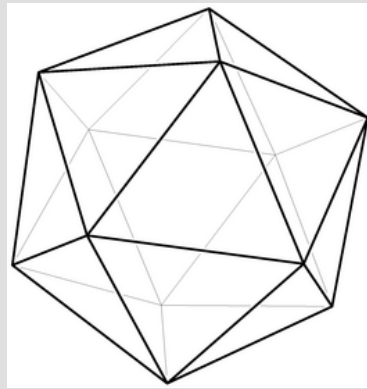
The cutest proofs I have seen have been in algebra and in combinatorics or discrete mathematics. So, today I will show you some nice and easy proofs that only will use some imagination and counting.

Note: By discrete mathematics I mean the study of mathematical structures that do not require the notion of continuity. Most of the objects studied in this area are countable sets.

Discrete mathematics usually includes set theory, combinatorics, graph theory, theory of computability and complexity, elementary probability theory and Markov chains, linear algebra, etc.

The Euler characteristic for polyhedrons

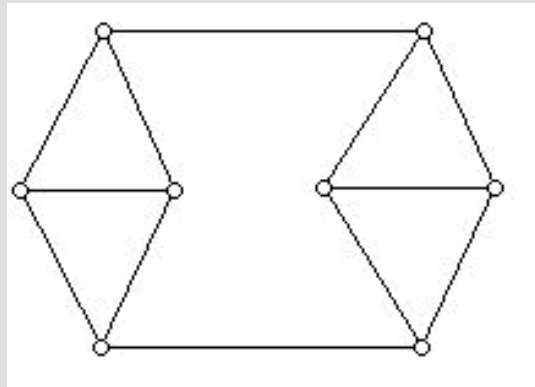
It is known that the number of vertices (V), edges (E) and faces (F) of a polyhedron are related by the Euler characteristic:



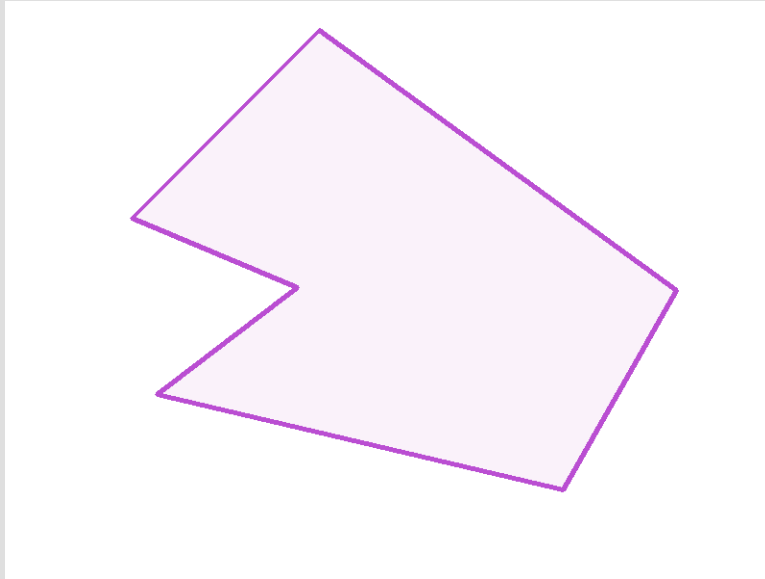
$$V-E+F=2$$

Consider a polyhedron P , think of it as a rubber sphere with flat sides. So, if we remove a point from one of the faces of P we can deform and flatten the solid to get a planar figure with vertices and edges that connect them (Oh! I love hand-waving).

Note that for any two vertices there is always a path made out of edges that joins them... Let's call this object the planar connected graph associated to P .



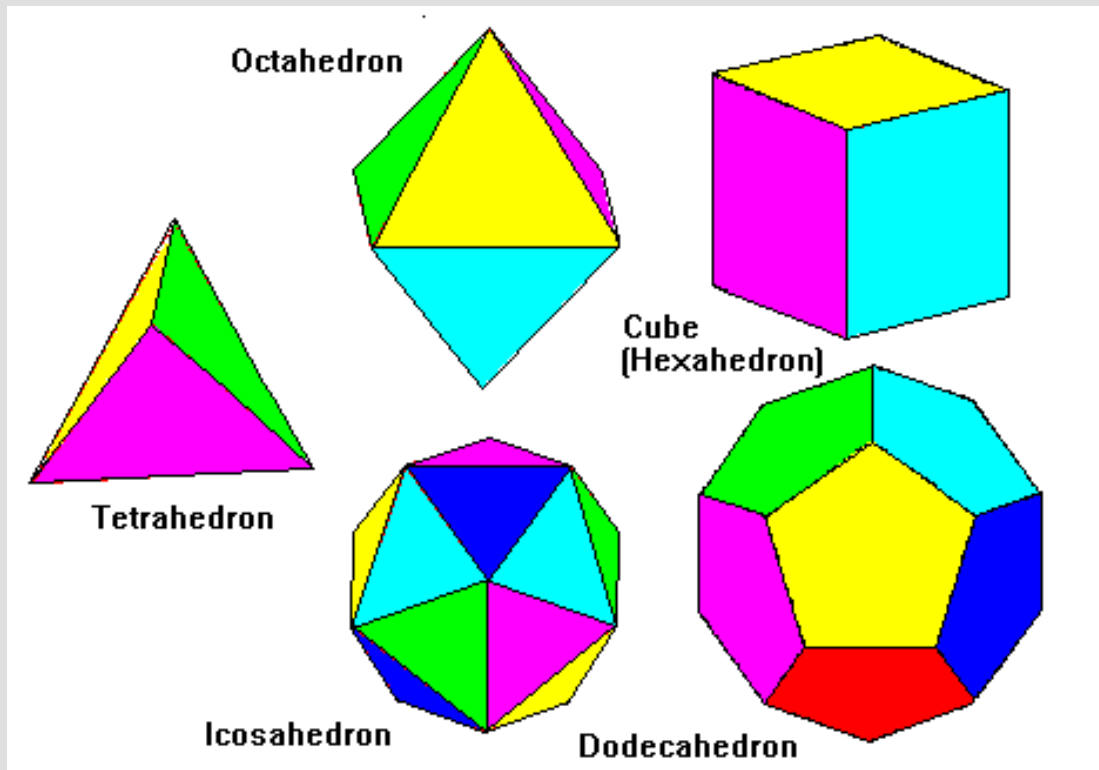
It is easy to see that by extracting an edge from the graph one is not only subtracting one edge but also a face, thus the Euler characteristic of the new figure is equal to the previous figure's. If one keeps going this way and is careful with some special situations (Oscar, draw something!) then, eventually, one gets to the point where the figure has been transformed into a polygon.



We now compute the Euler characteristic of this figure; we get $V - E + F = 2$ because the outside of the polygon is the face we pinched at the beginning. Hence, the Euler characteristic of the initial figure is also 2.

Euler characteristic and Platonic solids

We will use the Euler characteristic to show that there are only five Platonic solids (regular polyhedrons). They are:



Let P be a regular polyhedron, then each vertex has the same number of edges branching from it, call this number d (degree of the vertex). Also, each face has the same number of sides (edges), call this number k .

We know that each edge touches two vertices and that each vertex touches d vertices. Then, by counting the number of pairs (e, v) where e is an edge and v is a vertex we get:

$$2E = Vd$$

Similarly, since each edge separates two faces, then

$$Fk = 2E$$

By plugging these values into the Euler characteristic for P we get:

$$\frac{2E}{d} - E + \frac{2E}{k} = 2$$

or

$$E \left(\frac{2}{d} - 1 + \frac{2}{k} \right) = 2$$

We note that, since $E > 0$ then $\frac{2}{d} - 1 + \frac{2}{k} > 0$ or, in other words $\frac{2}{d} + \frac{2}{k} > 1$.

Since we need at least three sides per face then $k \geq 3$. Similarly, $d \geq 3$.

Now we see that if any d or k are larger than 5 then there is no way that the inequality is held. Thus d and k can be 3, 4, 5.

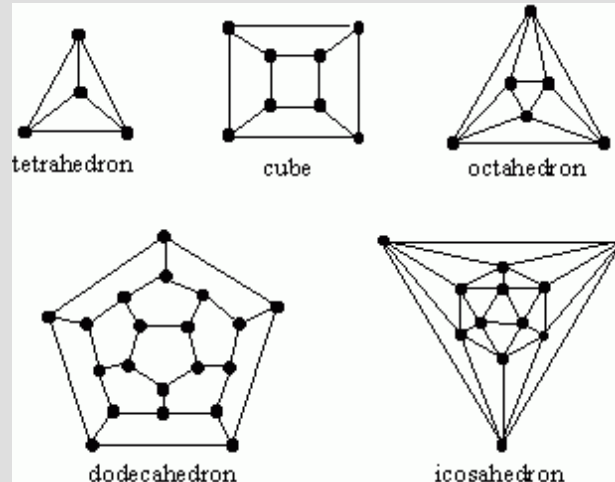
By just trying numbers we get that the solutions for the inequality are:

$$(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)$$

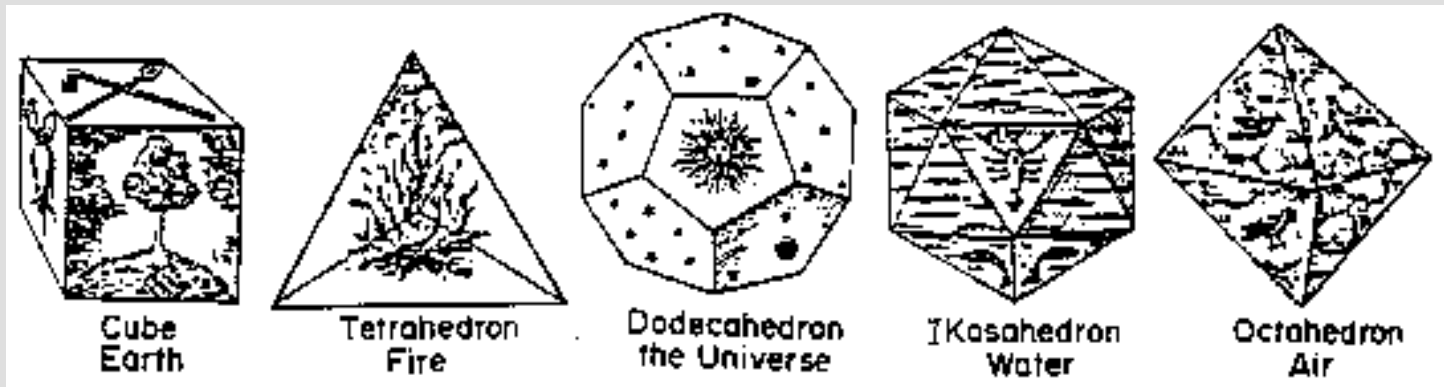
These pairs yield:

(d,k)	V	E	F	Name
$(3, 3)$	4	6	4	Tetrahedron
$(3, 4)$	12	8	6	Cube
$(4, 3)$	6	12	8	Octahedron
$(3, 5)$	20	30	12	Dodecahedron
$(5, 3)$	15	30	20	Icosahedron

Finally, the planar graphs associated to the Platonic solids are:



and the solids as drawn in Kepler's *Mysterium Cosmographicum*



Pick's theorem

Let $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. A basis of \mathbb{Z}^2 is a pair of linearly independent vectors e_1, e_2 so that the \mathbb{Z} -module generated by them is \mathbb{Z}^2 .

Let $\{e_1 = (a, b), e_2 = (c, d)\}$ be a basis of \mathbb{Z}^2 . Note that the parallelogram spanned by e_1, e_2 has area $A(e_1, e_2) = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|$.

Note that if $\{f_1 = (r, s), f_2 = (t, u)\}$ is another basis of \mathbb{Z}^2 then, there is an invertible matrix P so that $\begin{bmatrix} r & t \\ s & u \end{bmatrix} P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Since P has entries in \mathbb{Z} and is invertible, then its determinant is a unit in \mathbb{Z} .

It follows that $\left| \det \begin{bmatrix} r & t \\ s & u \end{bmatrix} \right| = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|$.

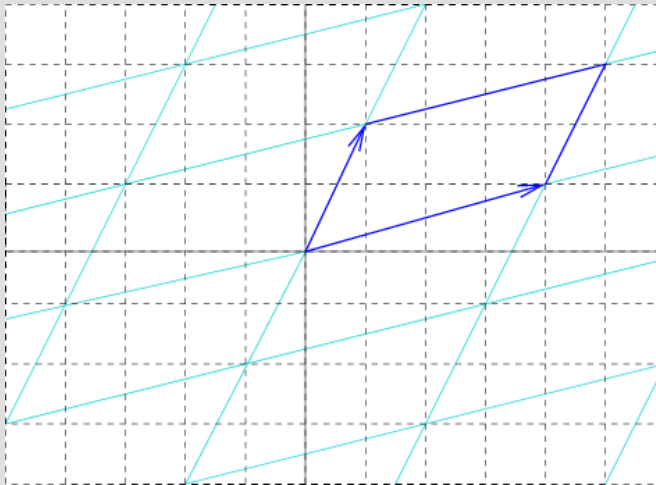
So, for any basis of \mathbb{Z}^2 the area of the associated parallelogram is the same, and equal to one because the canonical basis has a parallelogram with area 1.

Definition: A convex polygon $P \subset \mathbb{R}^2$ is called *elementary* if its vertices are points of \mathbb{Z}^2 (the vertices are integral) and P does not contain any other point of \mathbb{Z}^2 .

Lemma: Every elementary triangle Δ has area $\frac{1}{2}$.

Proof: Without loss of generality, assume that the vertices of Δ are $\{0, p_1, p_2\}$.

Let P be the parallelogram with vertices $0, p_1, p_2$ and $p_1 + p_2$



Note that the function $f : x \rightarrow (p_1 + p_2 - x)$ fixes P (f is a reflection across the origin and a translation afterwards) and also fixes \mathbb{Z}^2 . The important thing is that f sends \triangle to the other half of P . Hence, there are no integral points inside P .

Now note that the integral translates of P partition \mathbb{R}^2 (see picture above).

If some element $(x, y) \in \mathbb{Z}^2$ is not a vertex of one of these translations then this would imply that there is a point either inside or in the boundary of P that is not a vertex. This does not happen, thus all the $(x, y) \in \mathbb{Z}^2$ are vertices of the integral translations of P . It follows that $\{p_1, p_2\}$ is a basis of \mathbb{Z}^2 .

Then the area of P is one and the area of \triangle is $\frac{1}{2}$.

That was just a lemma for the big theorem.

Theorem (Pick): The area of a (not necessarily convex) polygon Q with integral vertices is given by

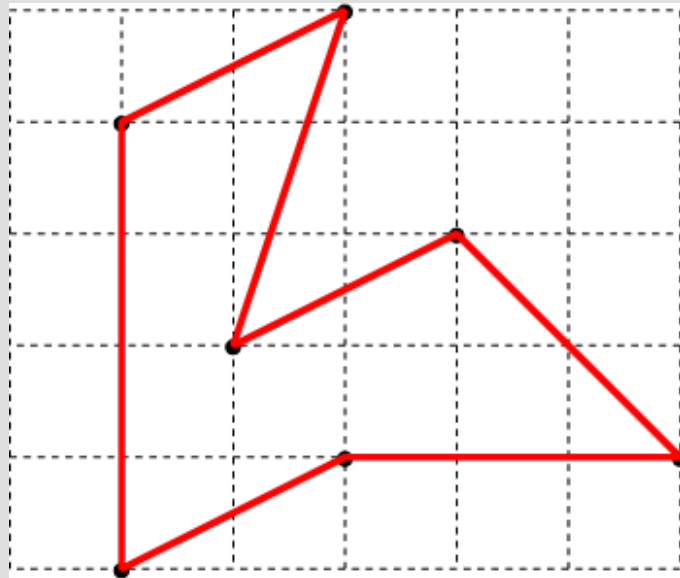
$$A(Q) = n_{int} + \frac{1}{2}n_{bd} - 1$$

where:

n_{int} = number of integral points in the interior points of Q

n_{bd} = number of integral points in the boundary of Q .

Example: For the following polygon,



$$n_{int} = 5 \quad n_{bd} = 13$$

Then, its area should be $5 + \frac{13}{2} - 1 = 10.5$... it is !!, by rearranging pieces of the polygon one can compute this area and get that number.

To prove this theorem we need to blindly believe that using ALL the integral points that are either interior or boundary points of the polygon we can triangulate Q .

At least we can do it in our example.

Assuming that, we use that each of the triangles used in the triangulation has to be elementary. It follows that the area of the polygon is $\frac{1}{2}T$ where T is the number of triangles in the triangulation.

Note that what we get after the triangulation is a planar connected graph and that the number of triangles in it is the number of faces of the graph minus one (the “exterior face”).

We know that the number of vertices in the graph is $n_{int} + n_{bd}$.

We now count the number of pairs (*edge, triangle*) with the edge being a side of the triangle, we get:

$$E_{bd} + 2E_{int} = 3T$$

where E_{bd} is the number of edges in the boundary of Q and E_{int} is the number of interior edges of the triangulation. Clearly $E_{bd} = n_{bd}$ so:

$$\begin{aligned} 3T &= n_{bd} + 2(E - n_{bd}) \\ &= 2E - n_{bd} \end{aligned}$$

It follows that $E = \frac{1}{2}(3T + n_{bd})$.

Plugging $V = n_{int} + n_{bd}$ and $E = \frac{1}{2}(3T + n_{bd})$ into Euler's formula we get:

$$\begin{aligned} 2 &= n_{int} + n_{bd} - \frac{1}{2}(3T + n_{bd}) + (T + 1) \\ &= n_{int} + \frac{1}{2}n_{bd} - \frac{1}{2}T + 1 \end{aligned}$$

so

$$\frac{1}{2}T = n_{int} + \frac{1}{2}n_{bd} - 1$$

Since the area of each triangle is $\frac{1}{2}$ then:

$$A(Q) = n_{int} + \frac{1}{2}n_{bd} - 1$$

THE END