Flat Flocks and Generalized j-planes

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- There exist a bijection $\xi : PG(n,K) \longrightarrow \mathcal{V}_n$.
- ξ maps the quadrics of $PG(n,K)$ onto the hyperplane sections of $\mathcal{V}_n$.

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The Segre variety $\mathcal{S}_{n,n}$ is the subset of $PG(m,K)$ containing all possible points $P_{\eta(A,B)}$.

- There is a bijection $\delta : PG(n,K) \times PG(n,K) \longrightarrow \mathcal{S}_{n,n}$

Generalizing flocks of $Q^+(3, q)$

Let $Q^+(3, q)$ denote the hyperbolic quadric of $PG(3, q)$, q any prime power. A flock of $Q^+(3, q)$ is a partition of the quadric in $q + 1$ irreducible conics (which are planar sections of $Q^+(3, q)$).

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Hence, it is natural to try to extend the idea of flocks of $Q^+(3, q)$ to partitions of $S_{n,n}$ into V_n's.

Flat flocks

Definition

A flat flock of $\mathcal{S}_{n,n}$ is a partition of $\mathcal{S}_{n,n}$ into Veroneseans obtained as linear sections of $\mathcal{S}_{n,n}$.

(A,B)-regular spreads and flat flocks

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Let A and B be two distinct members of a spread S of $PG(2n-1,q)$. We say S is (A,B)-regular if for every component $C \in S \setminus (A,B)$, the regulus generated by $\{A,B,C\}$ is contained in S.

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Theorem (Bader-Cossidente-Lunardon)

Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to (A,B)-regular spreads in $PG(2n-1,q)$. Moreover, the Veronese varieties of the partition correspond to $GF(q)$-reguli (reguli with $q+1$ lines).

Regulus hyperbolic covers and flat flocks

Definition

Let S be a spread in $PG(2n-1,q)$. A "regulus hyperbolic cover of order q" of S is a set of $(q^n-1)/(q-1)$ $GF(q)$-reguli that share two components of S and whose union is S.

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Theorem (Jha-Johnson)

Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to translation planes of order q^n that admit a regulus hyperbolic cover.

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Examples of flat flocks of $\mathcal{S}_{n,n}$ were found using the previous two theorems.

Generalized j-planes

Definition

A generalized j-plane is a translation plane with spread S in $PG(2n-1,q)$ given by the orbit of the subspace $y=x$ under the group

$$G = \left\{ \begin{bmatrix} f_M^{-1} & 0 \\ 0 & M \end{bmatrix};\ M \in M_n(q) \right\}$$

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Where $f : M_n(q) \rightarrow M_n(q)$ is a multiplicative function such that $f(M)^{q-1} = Id$ for all $M \in M_n(q)$.

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The class of all non-André generalized j-planes is a new family of translation planes.

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Corollary

New flat flocks can be constructed from non-André generalized j-planes.