$jj\ldots j$-planes

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Abstract

A new family of translation planes is constructed, it generalizes one found by Johnson, Pomareda and Wilke (J. Combin. Theory Ser. A 56 (1991), 272-284). Two more families of planes may be constructed from the one found by replacing some of the lines of these planes with different point-sets that play the role of lines. These processes will be explained in detail.

The basics

Consider a triple $\Pi = (P, L, \Im)$. Where P is a set of objects that we call points of Π, L is a subset of $\wp(P)$ that we call lines of Π and $\Im$ is a subset of $P \times L$ that we call Incidence.

We will say that a point p is incident with a line l, and that l is incident with p, if $(p, l) \in \Im$. We will also say that p is a point of l and that l contains p.

Two lines, l and m, intersect if there exist a point p that is incident with both. If the lines do not intersect, the lines are said to be parallel. This definition induces an equivalence relation in L.

Affine planes

Definition: $\Pi = (P, L, \Im)$ as above is an affine plane if it satisfies the following conditions:

i. There is a unique line incident with any two given points.

ii. For every line l and every point p not incident with l there is a unique line m incident with p and that does not intersect l.

iii. There are four points that are not in the same line.

The last condition rules out degenerate cases, it assures that our plane "looks" like a regular Euclidean plane.

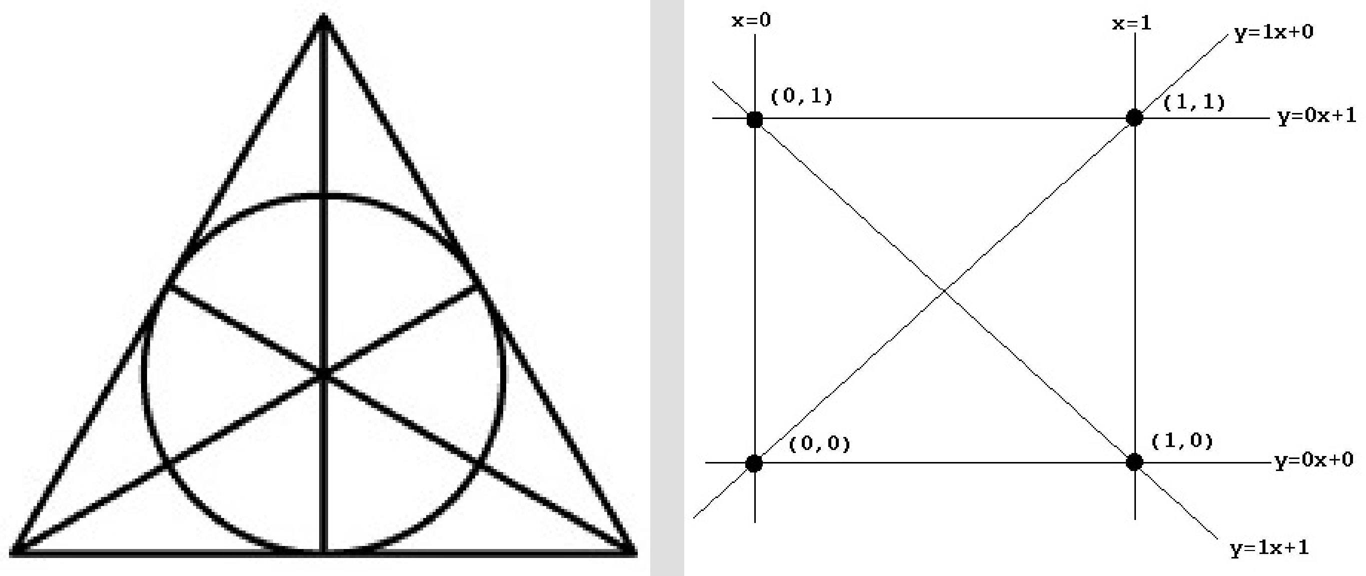

Fig. 1: The projective and affine planes over $\mathbb{F}_2$.

Order of a plane / Translation planes

For any finite affine plane Π there is a number n, called the order of the plane, so that the number of points in Π is n^2 and the number of lines is $n^2 + n$.
Also, there are n points per line and $n + 1$ lines per point.

All known finite affine planes have order p^h for p prime and $h \geq 1$.

We are interested in a subclass of affine planes that admit a group acting on the lines that fixes all parallel classes and that is transitive on the affine points of the plane. These planes are called *translation planes*.

Spreads

Definition: Let V be a $2n$-dimensional vector space over a field K. A spread S of V is a set of n-dimensional subspaces of V that intersect trivially and that partition the space.

The elements of S are called components of the spread S.

The direct sum of any two components of S is equal to V.

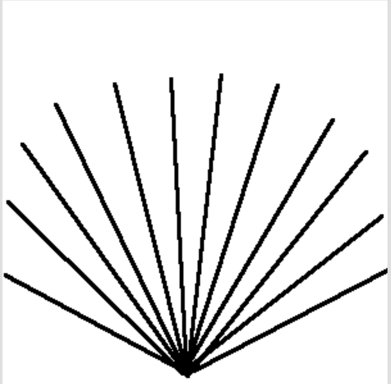

Fig. 2: Spread.

Relation between spreads and translation planes

Consider V, a $2n$-dimensional vector space over a field K, and let S be a spread of V. We define $\Pi = (P, L, \Im)$ by:

i. The elements of P are the points (vectors) of V

ii. The elements of L are the components of S and all its (additive) cosets.

iii. The incidence is given by the natural set theoretic inclusion.

Theorem: Π is an Affine Plane. Moreover, if $K = \mathbb{F}_q$ then, the order of Π is q^n.

Theorem: (André) Every translation plane can be represented by using a spread.

A little notation

Let V be a $2n$-dimensional vector space over K.

For a fixed matrix $M \in M_n(K)$, call $(y = xM)$, or simply ℓ_M, to the n-dimensional subspace

$$\{(x, y) \in V;\ y = xM\}\,.$$

Note that any n-dimensional subspace of V that is disjoint from $(x = 0)$ can be represented as $(y = xM)$, for some suitable $n \times n$ matrix M.

Example of a spread

Let $p(x) = x^3 - ax^2 - bx - c$ be irreducible in $\mathbb{F}_q[x]$.

It is not hard to see that the set E of all matrices of the form

$$M_{r,s,t} = \begin{bmatrix} r & s & t \\ ct & r+bt & s+at \\ c(s+at) & ct+b(s+at) & r+bt+a(s+at) \end{bmatrix},$$

where $r, s, t \in \mathbb{F}_q$, is a field of order q^3 contained in $GL(3,q) \cup \{0\}$.

Define $S = \{(x = 0)\} \cup \{(y = xM) \ ; \ M \in E\}$.

The fact that E is a field implies that S is a spread. Its associated translation plane has order q^3.

A more general field of matrices

Let $K = \{\alpha Id;\ \alpha \in \mathbb{F}_q\} \subset M_n(q)$.

Given a monic polynomial $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$, irreducible over $\mathbb{F}_q[x]$. Consider:

$$\theta = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

Note that θ is the companion matrix of $p(x)$, thus $p(\theta) = 0$. We define $F = K(\theta) \cong \mathbb{F}_{q^n}$.

Hence, $S = \{(x = 0)\} \cup \{(y = xM)\ ;\ M \in F\}$ is a spread with associated translation plane of order q^n.

jj...j-planes

Definition: Let $j_2, j_3, ..., j_n$ be fixed positive integers and let F be the field of the previous example. Define:

$$G = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix};\ M \in F^* \right\}$$

where $\Delta_M = diag(1, \partial^{j_2}, \partial^{j_3}, ..., \partial^{j_n})$ and $\partial = \det(M)$.

It is easy to see that G is a cyclic group of order $q^n - 1$.

In case $S = \{(x = 0), (y = 0)\} \cup O_G(y = x)$ is a spread, we say that its associated translation plane is a $j_2, j_3, ..., j_n$-plane, or simply a $jj...j$-plane.

A $00...0$-plane has the field F as associated spread. That is, a $00...0$-plane is Desarguesian.

A condition for existence

Lemma $S = \{(y=0)\} \cup O_G(y=x)$ is a spread, if and only if,

$$\det(\Delta_M M - Id) \neq 0 \;\; for\ every\ M \neq Id\ in\ F.$$

Actually, it is enough to check this for only the half of F, as

$$\det(\Delta_M M - Id) \neq 0 \iff \det(\Delta_{M^{-1}} M^{-1} - Id) \neq 0.$$

Lemma Let Π be a $j_2 j_3 \ldots j_n - plane$ of order q^n, then

$$\gcd(nj_i + 1, q-1) = 1 \;\; for\ every\ i > 1.$$

Using these conditions and a computer we found all $jj\ldots j$-planes of order 4^3, 7^3, 3^4, 4^4 and 5^4. Also, we were able to prove that there are no $jj\ldots j$-planes of order neither 3^n (n odd) nor 5^3 but the Desarguesians.

Collineations

Definition A collineation of an affine plane Π is an injective map that preserves the incidence relation.

Definition An isomorphism between two planes Π_1 and Π_2 is a bijection from the points of Π_1 onto the points of Π_2 that preserves the incidence relation.

Definition If a collineation Ψ of Π fixes a line ℓ pointwise and all the lines through a point P setwise, then Ψ is called a perspectivity.

1) If $P \in \ell$, then Ψ is called and elation.

2) If $P \notin \ell$, then Ψ is called a homology.

In either case, P is called the center of Ψ and ℓ is called the axis of Ψ.

Replacement and derivation on planes

The processes of replacement and derivation of planes consist in removing a certain set of lines of the plane and replace them by other suitable set of subspaces. A new translation plane is obtained.

Not every plane is replaceable or derivable.

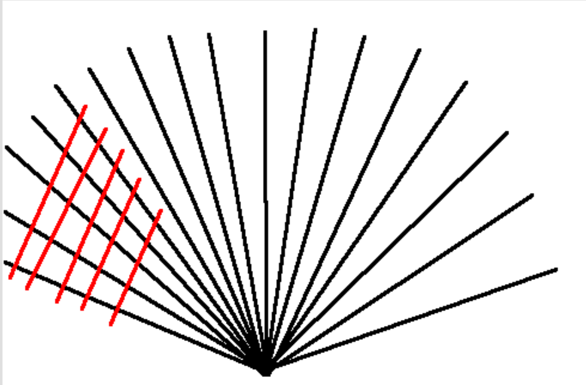

Fig. 2: Derivation in a plane.

Homology groups induce replacement and derivation

Note that G acts transitively on the non-zero components of S.

Also, its subgroup

$$\Gamma = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} \in G;\ det(M) = 1 \right\}$$

is a cyclic homology group with axis $(y = 0)$ and center (∞). Its order is $(q^n - 1)/(q - 1)$.

There is a second homology group of order $q - 1$ with axis $(x = 0)$ and center (0).

The orbits of Γ are replaceable sets of lines and, under the right conditions, induce derivable sets of lines.

There are new planes

For q any power of a prime and n dividing $q-1$, we have constructed at least one non-Desarguesian $jj...j$-plane of order q^n.

Unfortunately (for me), these planes were already known... André.

On the other hand, after comparing the $jj...j$-planes of orders 4^3, 7^3, 3^4, 4^4 and 5^4 found before with the known classes of translation planes, we have shown that there are new planes having order 4^3, 7^3, 3^4, 4^4 and 5^4. Moreover, each one of these new planes yield more new planes by replacement and derivation (when possible).

Full collineation group of a $jj...j$-plane

The translation complement of a non-André $jj...j$-plane Π fixes the lines $(x = 0)$ and $(y = 0)$. Moreover, the linear part of the translation complement of a $jj...j$-plane Π is isomorphic to the product of G and $\Gamma L(1, q^n)$.

Again, by comparing the collineation group of $jj...j$-planes with the collineation groups of known translation planes, we may say:

Theorem If a $jj...j$-plane does not admit a collineation interchanging $(x = 0)$ and $(y = 0)$ then the plane is new. On the other hand, if such a collineation exists, then the $jj...j$-plane is André.

Other technical results

1) A $j_2j_3...j_n$-plane of order q^n is Desarguesian, if and only if, $(j_2, ..., j_n) = (0, ..., 0)$.

2) Let Π be a $jj...j$-plane. Π is isomorphic to its transposed plane. However, $jj...j$-planes are not symplectic.

3) Replaced planes admit cyclic homology groups of order $(q^n - 1)/(q - 1)$. These planes also have spreads in $PG(2n - 1, q)$.

4) Whenever $q^n = h^2$ and $h - 1$ divides $(q^n - 1)/(q - 1)$, the (replaced) $jj...j$-plane is derivable.

Derived planes do not necessarily have spreads in $PG(2n - 1, q)$.

Flat flocks induced by $jj...j$-planes

Both $jj...j$-planes and replaced $jj...j$-planes admit affine homology groups of order $q-1$. The component orbits union the axis and coaxis are $\mathbb{F}_q$-reguli. Hence, we have a regulus hyperbolic cover. Each such regulus hyperbolic cover produces a partition of the Segre variety $S_{n,n}$ by Veronesians.