Flat flocks

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References

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2. Laura Bader, Antonio Cossidente and Guglielmo Lunardon. Generalizing flocks of $Q^+(3,q)$. Adv. Geom. 1 (2001), 323331.

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4. A generalization of j-planes... my PhD thesis.

Translation planes and spreads

We are interested in a subclass of affine planes that admit a group acting on the lines that fixes all parallel classes and that is transitive on the affine points of the plane. These planes are called *translation planes*.

Definition: Let V be a $2n$-dimensional vector space over a field K. A spread S of V is a set of n-dimensional subspaces of V that intersect trivially and that partition the space.

The elements of S are called components of the spread S.

The direct sum of any two components of S is equal to V.

Relation between spreads and translation planes

Consider V, a $2n$-dimensional vector space over a field $K \cong GF(q)$, and let S be a spread of V. We define $\Pi = (P, L, \Im)$ by:

i. The elements of P are the points (vectors) of V

ii. The elements of L are the components of S and all its (additive) cosets.

iii. The incidence is given by the natural set theoretic inclusion.

Theorem: Π is an affine plane. Moreover, the order of Π is q^n.

Theorem: (André / Bruck-Bose) There is a one-to-one correspondence between spreads of projective space and translation planes

Veronese varieties

The Veronesean variety of all quadrics of $PG(n,K)$, $n \geq 1$, is the variety

$$\mathcal{V}_n = \{(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, \ldots, x_0x_n, x_1x_2, \ldots, x_{n-1}x_n);\ (x_0, x_1, \ldots, x_n) \in PG(n,K)\}$$

of $PG(N,K)$ with $N = (n+1) + \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - 1 = n(n+3)/2$.

Considering the points of $PG(N,K)$ as generated by

$$(y_{00}, y_{11}, \ldots, y_{nn}, y_{01}, \ldots, y_{0n}, y_{12}, \ldots, y_{1n}, \ldots y_{n-1\ n})$$

then $\mathcal{V}_n$ is the intersection of all quadrics

$$F_{ij} = y_{ij}^2 - y_{ii}y_{jj} \quad and \quad F_{abc} = y_{aa}y_{bc} - y_{ab}y_{ac},$$

where $i \neq j$, $a \neq b \neq c \neq a$ and $i, j, a, b, c \in \{0, 1, \ldots, n\}$.

A few properties of $\mathcal{V}_n$

1. $\xi : PG(n,K) \longrightarrow PG(N,K)$ defined by

$$\xi(x_0, ..., x_n) = (y_{00}, ..., y_{n-1\ n}),$$

with $y_{ij} = x_i x_j$, is a bijection from $PG(n,K)$ onto $\mathcal{V}_n$.

2. $|\mathcal{V}_n| = |PG(n,K)|$.

3. The quadrics of $PG(n,K)$ are mapped by ξ onto all hyperplane sections of $\mathcal{V}_n$.

4. No hyperplane contains $\mathcal{V}_n$.

5. $\mathcal{V}_n$ is a cap of $PG(N,K)$.

6. For Π_s, an s-subspace of $PG(n,K)$, $\xi(\Pi_s) = \mathcal{V}_s$. Moreover, any $\mathcal{V}_s \subset \mathcal{V}_n$ has that form, for some $\Pi_s \subset PG(n,K)$.

7. $\mathcal{V}_1$ is a conic of $PG(2,K)$.

8. Lines are mapped to conics in $\mathcal{V}_n$

Segre varieties

Consider two projective spaces $PG(n_1, K)$ and $PG(n_2, K)$ with $n_i \geq 1$.

Let η be a bijection between $\{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\}$ and $\{0, 1, \ldots, m\}$, with $m + 1 = (n_1 + 1)(n_2 + 1)$.

The Segre variety of the 2 given projective spaces is the variety

$$\mathcal{S}_{n_1,n_2} = \left\{(x_0, x_1, \ldots, x_m);\ x_{\eta(i_1,i_2)} = x^{(1)}_{i_1} x^{(2)}_{i_2}\ \textit{with}\ (x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{n_i}) \in PG(n_i, K)\right\}$$

of $PG(m, K)$.

Let us see an example of this.

A few properties of $\mathcal{S}_{n_1,n_2}$

1. $\mathcal{S}_{n_1,n_2}$ is the intersection of all quadrics

$$x_{\eta(i_1,1_2)}x_{\eta(j_1,j_2)} - x_{\eta(i_1,j_2)}x_{\eta(j_1,1_2)}$$

2. $\delta : PG(n_1, K) \times PG(n_2, K) \longrightarrow \mathcal{S}_{n_1,n_2}$ defined by

$$(x_0^{(1)}, ..., x_{n_1}^{(1)}) \times (x_0^{(2)}, ..., x_{n_2}^{(2)}) \mapsto (x_0, ..., x_m),$$

where $x_j = x_{i_1}^{(1)} x_{i_2}^{(2)}$, is a bijection.

3. $|\mathcal{S}_{n_1,n_2}| = |PG(n_1, K)||PG(n_2, K)|$.

4. $\mathcal{S}_{n,n} \cap \Pi_{n(n+3)/2} = \mathcal{V}_n$.

Flocks of $Q^+(3,q)$

Let $Q^+(3,q)$ denote the hyperbolic quadric of $PG(3,q)$, q any prime power. A flock of $Q^+(3,q)$ is a partition of the quadric in $q+1$ irreducible conics. A flock is linear if all the planes of the conics of the flock contain a common line.

Flocks of $Q^+(3,q)$ have been classified for q even, and it was proven that they are necessarily linear.

For q odd, the translation plane associated with a flock of $Q^+(3,q)$ is coordinatized by a nearfield, this helped to obtain a complete classification of the flocks of $Q^+(3,q)$, which are either linear, or of Thas type (obtained by taking two halves of suitable linear flocks), or exceptional (existing for $q = 11, 23, 59$).

How do we get the translation plane associated to a flock of $Q^+(3,q)$?

Generalizing flocks of $Q^+(3, q)$

As $Q^+(3, q)$ is the smallest Segre variety, a conic in $PG(2, q)$ is the smallest Veronesean, and the Klein quadric is the Grassmannian of the lines of $PG(3, q)$, we can try to extend the notion of flock to the Segre variety $S_{n,n}$, studying it via the Grassmannian $\mathcal{G}_{1,2n+1}$.

Note that $|\mathcal{S}_{n,n}| = |PG(n, q)|^2$ and $|\mathcal{V}_n| = |PG(n, q)|$, then a partition of $\mathcal{S}_{n,n}$ into $\mathcal{V}_n$'s has to be done using $(q^n - 1)/(q - 1)$ Veroneseans.

Flat flocks

A flock of $\mathcal{S}_{n,n}$ is a partition of it into $(q^n-1)/(q-1)$ caps of size $(q^n-1)/(q-1)$.

If the caps are Veronesean varieties obtained as sections of $\mathcal{S}_{n,n}$ by linear subspaces of the projective space $PG(n^2+2n,q)$ in which $\mathcal{S}_{n,n}$ resides, then the flock is called a flat flock.

The flat flock is linear if all the subspaces of its Veronesean members share an n-dimensional subspace of $PG(n^2+2n,q)$.

As in the $Q^+(3,q)$ case, we want to relate flat flocks with a class of spreads.

(A,B)-regular spreads and flat flocks

Definition Let A and B be two distinct members of a spread S of $PG(2n-1, q)$. We say S is (A, B)-regular if for every component $C \in S \setminus (A, B)$, the regulus generated by $\{A, B, C\}$ is contained in S.

Theorem Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to (A, B)-regular spreads in $PG(2n-1, q)$. Moreover, the Veronese varieties of the partition correspond to $GF(q)$-reguli (reguli with $q+1$ lines).

Hyperbolic covers and flat flocks

Let S be a spread in $PG(2n-1,q)$. A "regulus hyperbolic cover of order q" of S is a set of $(q^n-1)/(q-1)$ $GF(q)$-reguli that share two components of S and whose union is S.

Theorem Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to translation planes of order q^n that admit a regulus hyperbolic cover.

Some examples of flat flocks have been found. They are related to planes that are Desarguesian, semifield, regular nearfield $N(n+1,q)$ or André.

A little notation

Let V be a $2n$-dimensional vector space over K.

For a fixed matrix $M \in GL(n, K)$, call $(y = xM)$, or simply ℓ_M, to the n-dimensional subspace

$$\{(x, y) \in V;\ y = xM\}\,.$$

Note that any n-dimensional subspace of V that is disjoint from $(x = 0)$ can be represented as $(y = xM)$, for some suitable M.

A field of matrices

Let $K = \{\alpha Id;\ \alpha \in GF(q)\} \subset M_n(q)$.

Given a monic polynomial $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$, irreducible over $GF(q)[x]$. Consider:

$$\theta = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

Note that θ is the companion matrix of $p(x)$, thus, $p(\theta) = 0$. We define $F = K(\theta) \cong GF(q^n)$.

Hence, $S = \{(x = 0)\} \cup \{(y = xM)\ ;\ M \in F\}$ is a spread with associated translation plane of order q^n.

jj...j-planes

Definition: Let $j_2, j_3, ..., j_n$ be fixed positive integers and let F be the field of the previous example. Define:

$$G = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix}; \; M \in F^* \right\}$$

where $\Delta_M = diag(1, \partial^{j_2}, \partial^{j_3}, ..., \partial^{j_n})$ and $\partial = \det(M)$.

It is easy to see that G is a cyclic group of order $q^n - 1$.

Now consider the n-dimensional subspace of V given by the formula $y = x$. In case $S = \{(y = 0)\} \cup O_G(y = x)$ is a spread we will say that its associated plane is a $j_2, j_3, ..., j_n$-plane, or simply a $jj...j$-plane.

Groups acting on $jj...j$-planes

The group G acts transitively on the non-zero components of S.

Also, there is a (homology) subgroup

$$\Gamma = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} \in G;\ det(M) = 1 \right\}$$

of G, which fixes $(y = 0)$ pointwise and fixes (setwise) every line through (∞).

There is a second homology group of order $q-1$ that is induced by

$$\Omega = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} \in G;\ M = rId,\ r \in GF(q)^* \right\}$$

As a matter of fact, the group is

$$\Lambda = \left\{ \begin{bmatrix} r^{-1}\Delta_M^{-1} & 0 \\ 0 & Id \end{bmatrix} \in G;\ M = rId,\ r \in GF(q)^* \right\}$$

Note that Λ fixes $(x = 0)$ pointwise and fixes (setwise) every line through (0).

Corollary Every $jj...j$-plane and replaced $jj...j$-plane of order q^n induces a flat flock.

Proof Λ induces a regulus hyperbolic cover of the plane.