An invariant for spreads of $PG(3, q)$

Oscar Vega
California State University, Fresno

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$PG(3, q)$, spreads, parallelisms

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- A spread is a partition of the $(q+1)(q^2+1)$ points of $PG(3, q)$ into q^2+1 (disjoint) lines.
- A parallelism is a partition of the $(q^2+1)(q^2+q+1)$ lines into q^2+q+1 (disjoint) spreads.

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  for some $M \in M_2(q)$.
- If we assume that the line $x = 0$ belongs to the spread $\mathcal{S}$ then $\mathcal{S}$ can be thought of as a set of q^2 matrices.

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- A parallelism with deficiency one can always be extended to a 'full' parallelism.
- A very old conjecture says that a **regular** parallelism with deficiency one can only be extended to a **regular** parallelism.

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- Let L^* be the set of all lines disjoint from ℓ. Identify L^* with $M_2(q)$.
- Fix $M \in M_2(q)$. Consider the characteristic function of M,

$$\chi_M : M_2(q) \to \mathbb{F}_q \ , \qquad \chi_M(N) = \begin{cases} 1 & \text{for} \quad M = N \\ 0 & \text{for} \quad M \neq N \end{cases}$$

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- Using χ_M we define the characteristic function of a line of L^*. Also, we naturally extend χ to any $A \subset L^*$.

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- From now on, we will consider all characteristic functions to be polynomials as described above.

The degree of a spread

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- Note that the polynomial and the degree associated to a spread depend on the choosing of ℓ and m.

Example

- For $\theta \neq \square$ in $\mathbb{F}_q$, consider the regular spread

$$S = \left\{ \begin{bmatrix} u & \theta t \\ t & u \end{bmatrix} ; \ t, u \in \mathbb{F}_q \right\}$$

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- Hence, $deg(S) = 2(q - 1)$.

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- Note that the previous result does not mention the line m.
- Actually, the result implies that if $deg(S) = 2(q-1)$ with respect to one line ℓ, then $deg(S) = 2(q-1)$ with respect to any line $\tilde{\ell} \in S$.

Invariance

- Consider the group Ω that stabilizes ℓ (which also leaves L^* invariant).

$$\Omega = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix};\ A, B, D \in M_2(q),\ \det(AD) \neq 0 \right\}.$$

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- Let $U, V \subset L^*$, and $\Psi \in \Omega$, then

$$deg\left(\chi_{\Psi(U)}\right) = deg\left(\chi_U\right).$$

and

$$deg\left(\chi_{\Psi(U)} - \chi_{\Psi(V)}\right) = deg\left(\chi_U - \chi_V\right).$$

Back to the parallelisms

- Let $P^- = \{S_1, \ldots, S_{q^2+q}\}$ be a regular partial parallelism with deficiency one, and let S be the spread that extends P^- to a parallelism. Then, S has degree $\leq 3(q-1)$ with respect to any of its lines.

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- If the unknown spread is (André) subregular of index t, then $t < (q-1)/2$. In particular, if $q \neq 3$ then the unknown spread is not Hall.