Groups of derangements of the $n$-cube.

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The $n$-cube.

Definition

The $n$-dimensional cube $Q_n$, AKA $n$-cube, is the graph with

(i) vertices $y = (y_1, y_2, \ldots, y_n)$, where $y_i = \pm 1$ for all $i = 1, 2, \ldots, n$,

(ii) edges joining any two vertices that differ in exactly one component.

Example: $Q_3$
**Definition**

A $k$-face $F$ of $Q_n$ is a $k$-subcube whose vertices have $n - k$ of the coordinates predetermined. That is,

$$V(F) = \{ y = (y_1, \ldots, y_n) \in Q_n; \; y_{i_1} = a_{i_1}, \ldots, y_{i_{n-k}} = a_{i_{n-k}} \},$$

where, of course, each $a_{ij} = \pm 1$.

The edges of $F$ are inherited from the edges of $Q_n$.

**Examples:**

- **1-face of $Q_3$**
  - $(1,1,1)$
  - $(-1,1,1)$
  - $(1,-1,1)$
  - $(-1,-1,1)$
  - $(*,1,-1)$

- **2-face of $Q_3$**
  - $(1,1,1)$
  - $(-1,1,1)$
  - $(1,-1,1)$
  - $(-1,-1,1)$
  - $(*,1,*)$
Automorphisms of $Q_n$.

- The automorphism group of the cube is $B_n = S_n \wr \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$. This group is sometimes called the hyperoctahedral group.

- We denote the elements in $B_n$ by $(\sigma; x)$, where $\sigma \in S_n$ and \(x = (x_1, x_2, \cdots, x_n) \in (\mathbb{Z}_2)^n\). The multiplication is given by

\[
(\sigma; x)(\tau; y) = (\sigma \tau; x^\tau y)
\]

where $x^\tau = (x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(n)})$, and $x^\tau y$ is computed by component to component multiplication.

- The action of $B_n$ on $Q_n$ is given by $(\sigma, x)y = xy^\sigma$. 
Derangements of $Q_n$. 

**Definition**

1. A derangement of the $k$-faces of $Q_n$ is an element of $B_n$ that acts freely on the set of all $k$-faces of $Q_n$.

2. A group $G$ will be called a derangement of the $k$-faces of $Q_n$ if it is isomorphic to a subgroup $H$ of $B_n$ such that every non-identity element in $H$ is a derangement of the $k$-faces of $Q_n$. In such a case we write

$$G \vdash_k B_n.$$ 

**Theorem (Cusick)**

If $G$ is a finite group and $G \vdash_k B_n$ for some $n \geq 1$, then $\gcd(k, |G|) = 2^s$ for some $s \geq 0$. 

The Problem.

Question
Let $G$ be a finite group and $k$ be such that $\gcd(k, |G|) = 2^s$ for some $s \geq 0$. Is there an $n$ such that $G \models kB_n$?

The first author (Cusick) proved that the answer to this question is yes if:

- $|G|$ is odd, or
- $|G| = 2^s$, for some $s$, or
- $G \cong \mathbb{Z}_m$, for some $m$.

Remark
Most of these results are proved using the Chen-Stanley criterion and outer products.
The Chen-Stanley Criterion & Outer Products

**Definition**

If \( \sigma = (i_1 i_2 \ldots i_s) \) is a cycle in \( S_n \) and \( x \in (\mathbb{Z}_2)^n \), then \( x_\sigma = x_{i_1}x_{i_2} \cdots x_{i_s} \).

**Theorem (Chen-Stanley, 1993)**

A symmetry \((\pi; x) \in B_n\) is a derangement of the set of \( k \)-faces in \( Q_n \) if, and only if, for every \( k \)-element \( \pi \)-invariant subset \( I \subset \{1, \ldots, n\} \), \( x_\sigma = -1 \) for some cycle \( \sigma \) in \( \pi \) disjoint from \( I \).

**Definition**

The outer product \( \times : B_n \times B_m \to B_{n+m} \) is defined by \((\pi; x) \times (\theta; y) = (\pi \times \theta; x, y)\), where \( \pi \times \theta \) is the permutation given by

\[
\pi \times \theta = \begin{pmatrix}
1 & 2 & \cdots & n & n+1 & \cdots & n+m \\
\pi(1) & \pi(2) & \cdots & \pi(n) & n+\theta(1) & \cdots & n+\theta(m)
\end{pmatrix}
\]
Main Theorem.

Our main theorem is that the answer to our question is always yes. That is

**Theorem (C-V)**

Let $G$ be a finite group and $k$ be such that $\gcd(k, |G|) = 2^s$ for some $s \geq 0$. Then, there is an $n$ such that $G$ is a derangement of the $k$-faces of $Q_n$.

The proof is (almost) constructive:

(i) Get a representation $\rho$ of the 2-Sylow of $G$ into some $B_n$ that is ‘good’
(ii) Get the induced representation of $\rho$ for $G$. Prove it is ‘good’.
(iii) Use outer products to get a representation of $G$ into some $B_m$ that satisfies the Chen-Stanley criterion.
Thank you!

Any questions?