

Groups of derangements of the n -cube.

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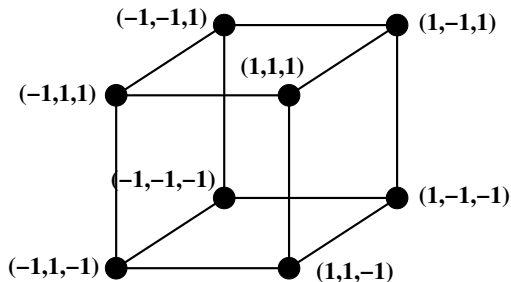
The n -cube.

Definition

The n -dimensional cube Q_n , AKA n -cube, is the graph with

- (i) vertices $\mathbf{y} = (y_1, y_2, \dots, y_n)$, where $y_i = \pm 1$ for all $i = 1, 2, \dots, n$,
- (ii) edges joining any two vertices that differ in exactly one component.

Example: Q_3



k -faces of Q_n .

Definition

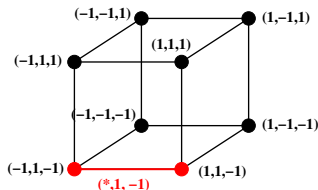
A k -face F of Q_n is a k -subcube whose vertices have $n - k$ of the coordinates predetermined. That is,

$$V(F) = \{\mathbf{y} = (y_1, \dots, y_n) \in Q_n; y_{i_1} = a_{i_1}, \dots, y_{i_{n-k}} = a_{i_{n-k}}\},$$

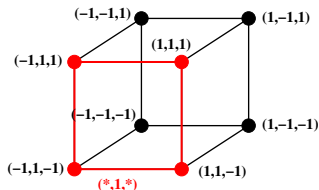
where, of course, each $a_{i_j} = \pm 1$.

The edges of F are inherited from the edges of Q_n .

1-face of Q_3



2-face of Q_3



Examples:

Automorphisms of Q_n .

- The automorphism group of the cube is $B_n = S_n \wr \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$. This group is sometimes called the hyperoctahedral group.
- We denote the elements in B_n by $(\sigma; \mathbf{x})$, where $\sigma \in S_n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in (\mathbb{Z}_2)^n$. The multiplication is given by

$$(\sigma; \mathbf{x})(\tau; \mathbf{y}) = (\sigma\tau; \mathbf{x}^T \mathbf{y})$$

where $\mathbf{x}^T = (x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(n)})$, and $\mathbf{x}^T \mathbf{y}$ is computed by component to component multiplication.

- The action of B_n on Q_n is given by $(\sigma, \mathbf{x})\mathbf{y} = \mathbf{xy}^\sigma$.

Derangements of Q_n .

Definition

- 1 A derangement of the k -faces of Q_n is an element of B_n that acts freely on the set of all k -faces of Q_n .
- 2 A group G will be called a derangement of the k -faces of Q_n if it is isomorphic to a subgroup H of B_n such that every non-identity element in H is a derangement of the k -faces of Q_n . In such a case we write

$$G \vdash_k B_n.$$

Theorem (Cusick)

If G is a finite group and $G \vdash_k B_n$ for some $n \geq 1$, then $\gcd(k, |G|) = 2^s$ for some $s \geq 0$.

The Problem.

Question

Let G be a finite group and k be such that $\gcd(k, |G|) = 2^s$ for some $s \geq 0$. Is there an n such that $G \vdash_k B_n$?

The first author (Cusick) proved that the answer to this question is yes if:

- $|G|$ is odd, or
- $|G| = 2^s$, for some s , or
- $G \cong \mathbb{Z}_m$, for some m .

Remark

Most of these results are proved using the Chen-Stanley criterion and outer products.

The Chen-Stanley Criterion & Outer Products

Definition

If $\sigma = (i_1 i_2 \dots i_s)$ is a cycle in S_n and $\mathbf{x} \in (\mathbb{Z}_2)^n$, then $x_\sigma = x_{i_1} x_{i_2} \dots x_{i_s}$.

Theorem (Chen-Stanley, 1993)

A symmetry $(\pi; \mathbf{x}) \in B_n$ is a derangement of the set of k -faces in Q_n if, and only if, for every k -element π -invariant subset $I \subset \{1, \dots, n\}$, $x_\sigma = -1$ for some cycle σ in π disjoint from I .

Definition

The *outer product* $\times : B_n \times B_m \rightarrow B_{n+m}$ is defined by

$(\pi; \mathbf{x}) \times (\theta; \mathbf{y}) = (\pi \times \theta; \mathbf{x}, \mathbf{y})$, where $\pi \times \theta$ is the permutation given by

$$\pi \times \theta = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & \dots & n+m \\ \pi(1) & \pi(2) & \dots & \pi(n) & n+\theta(1) & \dots & n+\theta(m) \end{pmatrix}$$

Main Theorem.

Our main theorem is that the answer to our question is always yes. That is

Theorem (C-V)

Let G be a finite group and k be such that $\gcd(k, |G|) = 2^s$ for some $s \geq 0$. Then, there is an n such that G is a derangement of the k -faces of Q_n .

The proof is (almost) constructive:

- (i) Get a representation ρ of the 2-Sylow of G into some B_n that is 'good'
- (ii) Get the induced representation of ρ for G . Prove it is 'good'.
- (iii) Use outer products to get a representation of G into some B_m that satisfies the Chen-Stanley criterion.

Thank you!

Any questions?