An Algebraic Combinatorial View To Knots And Links

Discrete Math Seminar UCDHSC DDC. January 29, 2007

“When Alexander reached Gordium, he was seized with a longing to ascend to the acropolis, where the palace of Gordius and his son Midas was situated, and to see Gordius’ waggon and the knot of the waggon’s yoke... Over and above this there was a legend about a waggon, that anyone who untied the knot of the yoke would rule Asia. The knot was of cornel bark, and you could not see where it began or ended. Alexander was unable to find how to untie the knot but unwilling to leave it tied, in case this caused a disturbance among the masses; some say that he struck it with his sword, cut the knot, and said it was now untied - but Aristobulus says that he took out the pole-pin, a bolt driven right through the pole, holding the knot together, and so removed the yoke from the pole. I cannot say with confidence what Alexander actually did about this knot, but he and his suite certainly left the waggon with the impression that the oracle about the undoing of the knot had been fulfilled, and in fact that night there was thunder and lightning, a further sign from heaven; so Alexander in thanksgiving offered sacrifice next day to whatever gods had shown the signs and the way to undo the knot.”

Lucius Flavius Arrianus, Anabasis Alexandri, Book II, c. 150 A.D.

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In fact, the first Topology book already mentions knots.

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The first systematic study of knots was done by Tietze, and followed some ideas and results by Poincaré. He associated a group to a knot (the fundamental group of the complement of the knot).

In 1987 it is shown that two knots are equivalent if their complements are homeomorphic.

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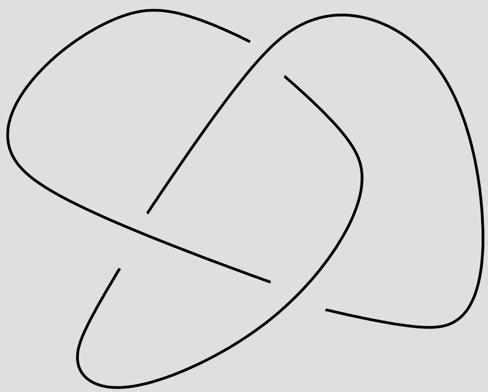

Fig. 1: Trefoil knot.

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A link is a set of disjoint knots. Each knot is a component of the link. In particular, a knot is a one-component link.

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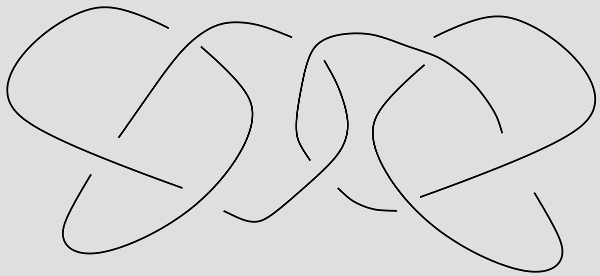

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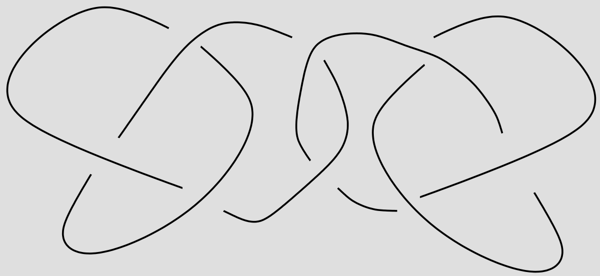

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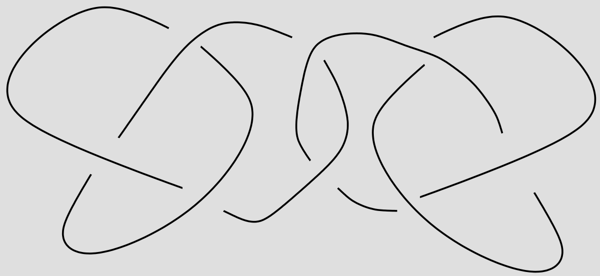

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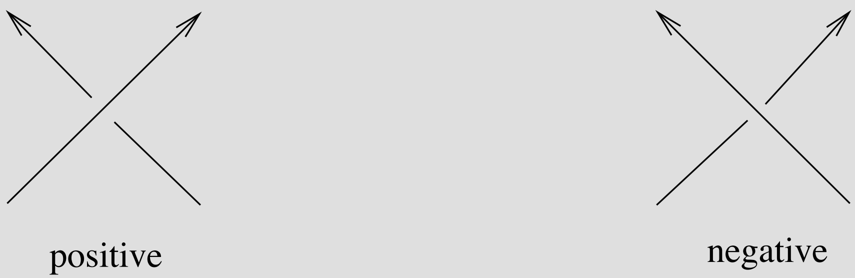
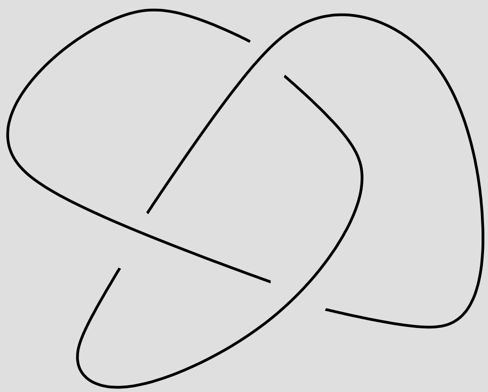


Representation of a knot

The easiest, and more natural, way to represent a knot is by using a projection on a plane. the result is a diagram with crossings that show what part of the knot goes over and under.



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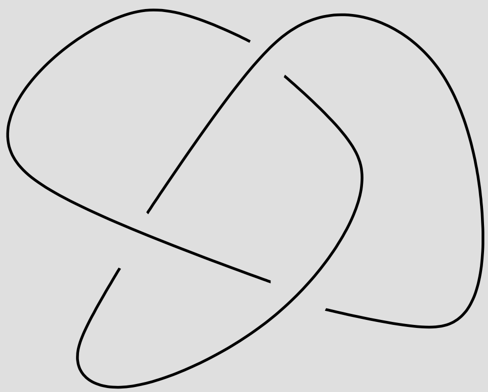


Fig. 3: Diagram of a knot.

The existence of a projection that does not leave questions on what the knot is in the space is a corollary of Sard's Theorem.

Representation of a knot

We could also, consider a diagram given by a polygon.

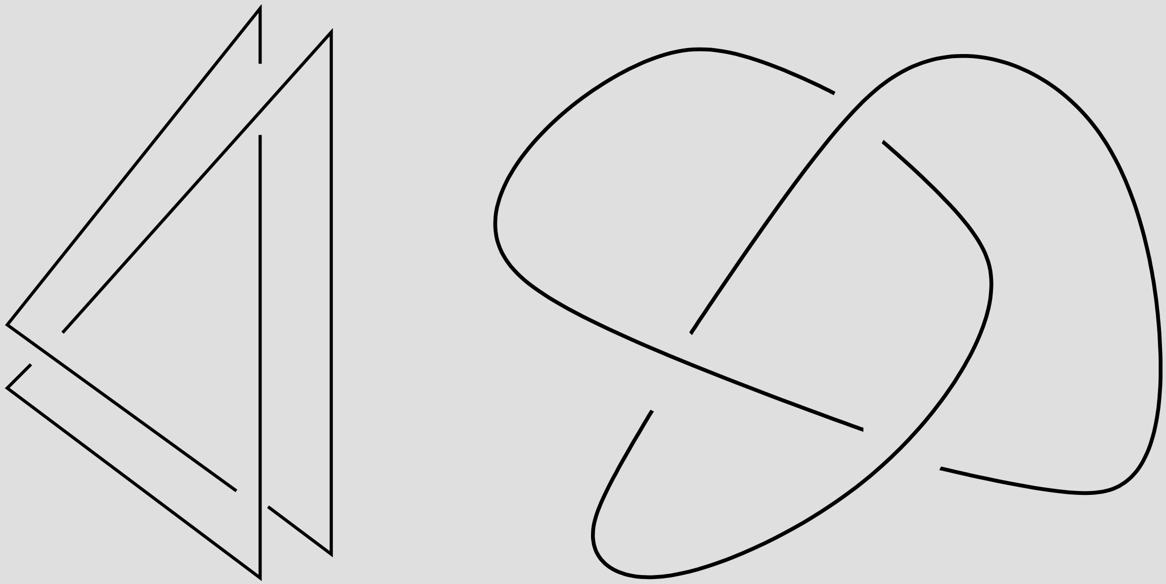


Fig. 4: Polygonal and regular diagram of a knot.

Representation of a knot

If the polygon may be chosen to have finitely many sides, we say the knot is tame, otherwise the knot is a wild knot.

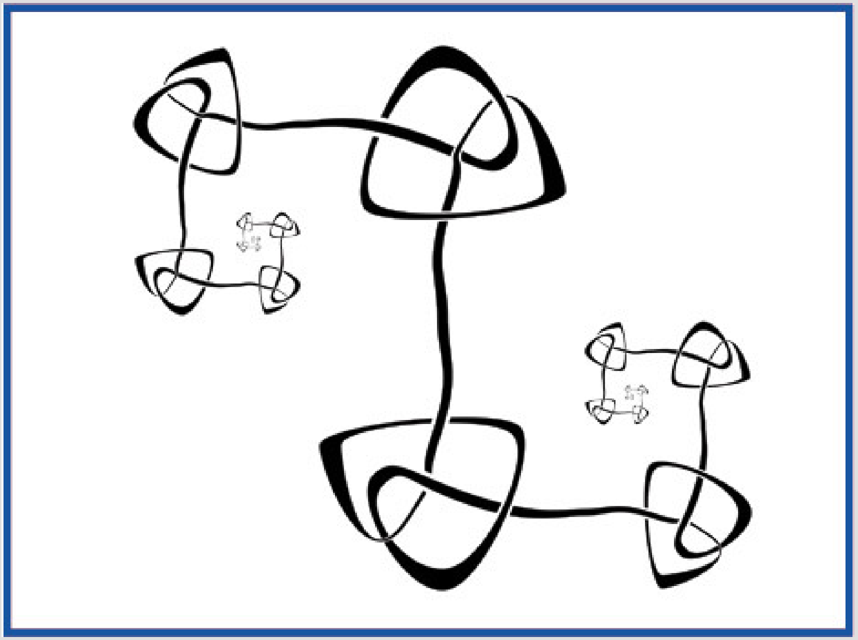


Fig. 5: A wild knot.

Examples of knots

The simplest non-trivial examples of knots are:

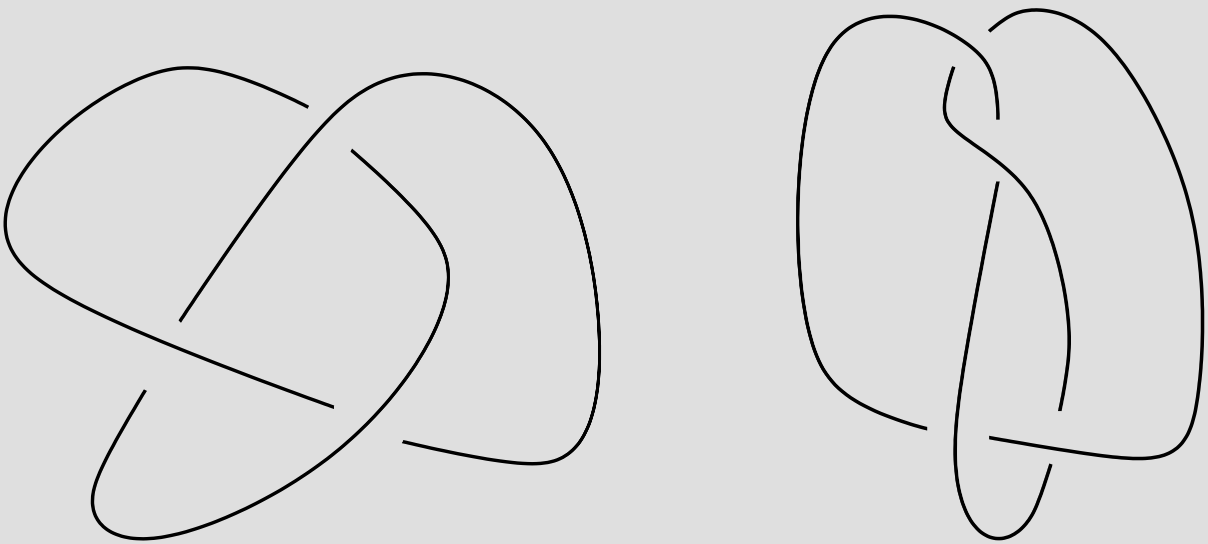


Fig. 6: Trefoil and figure 8 knots.

Equivalence of knots

Two knots K_1 and K_2 are equivalent if one can deform $\mathbb{R}^3$ in such a way that one starts with K_1 and ends up with K_2.

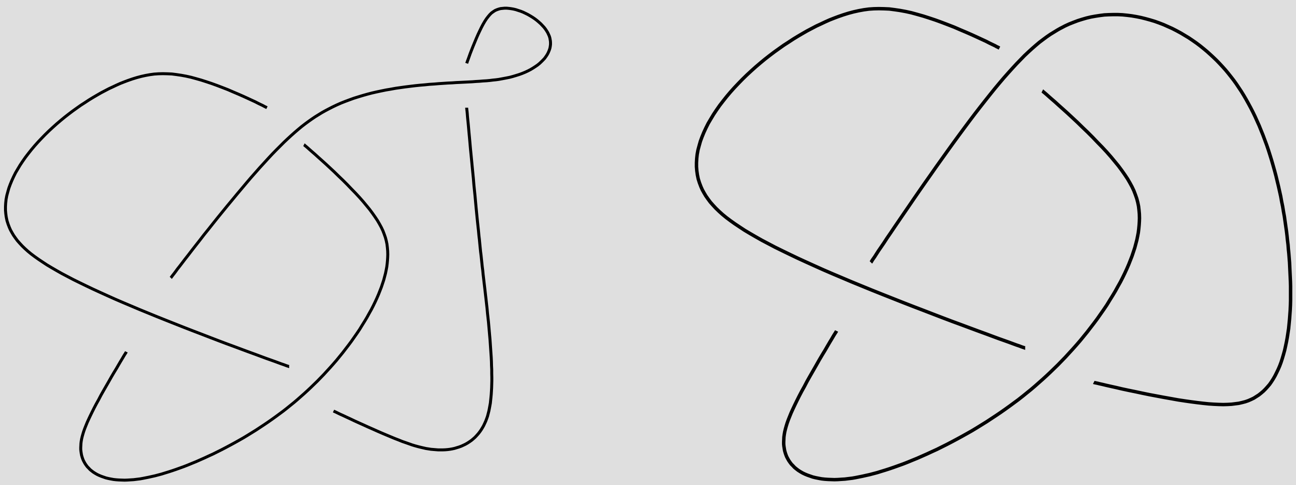


Fig. 7: Two equivalent knots.

Equivalence of knots

It is not always easy to determine whether or not two given knots are equivalent. For example, the following two knots are equivalent

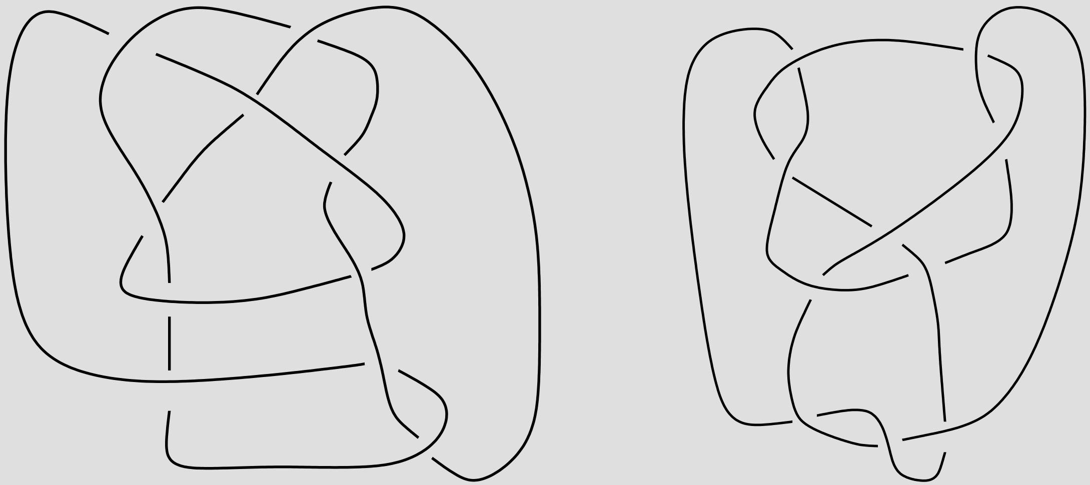
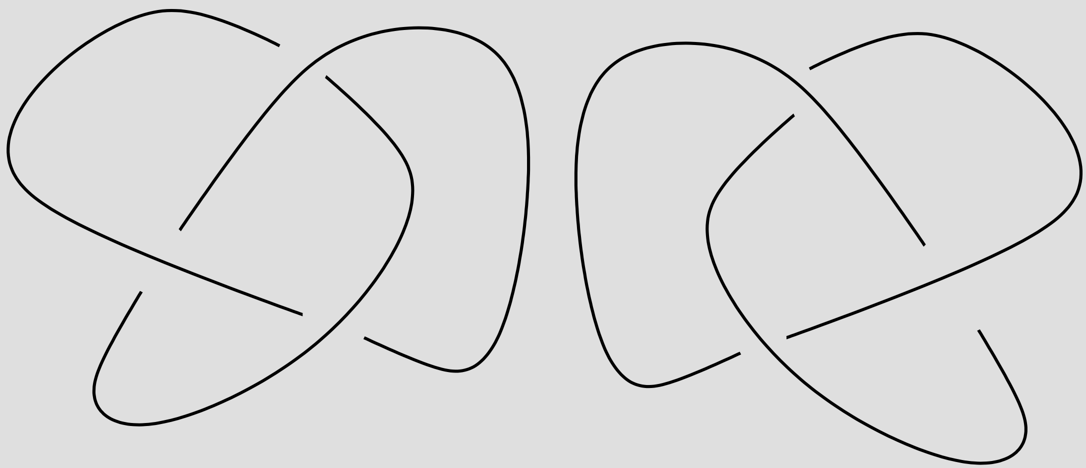


Fig. 8: Perko knots.

Equivalence of knots

Sometimes knots that seem to be very similar are not equivalent. For instance, the trefoil knot and its mirror image are not equivalent to each other. This was proved by Dehn.



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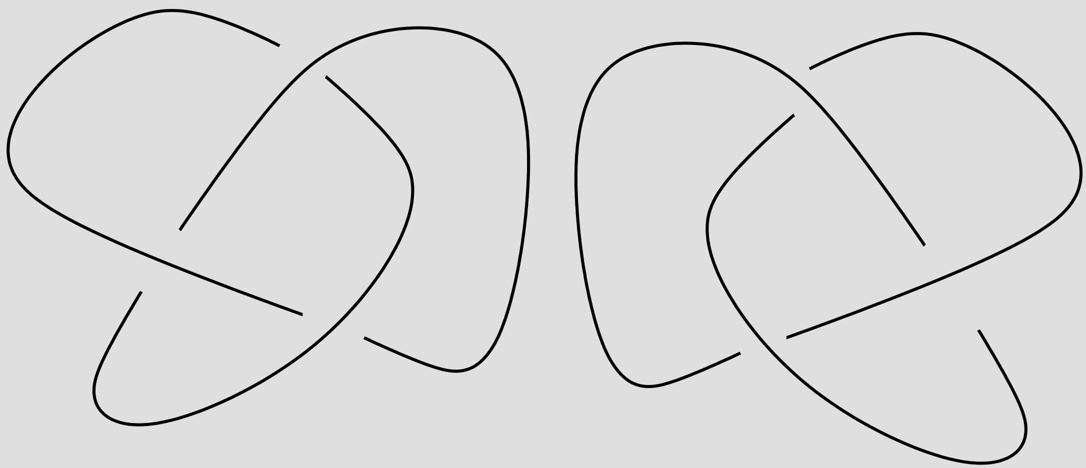

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On the other hand, the figure 8 knot is equivalent to its mirror image.

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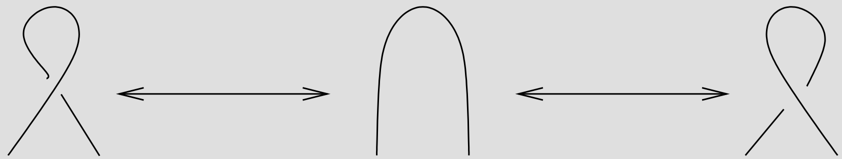

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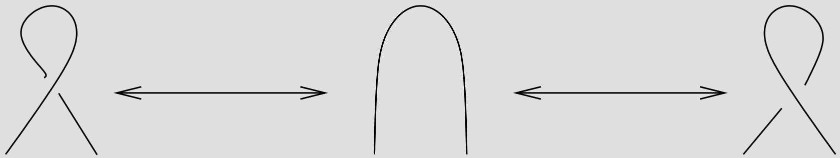

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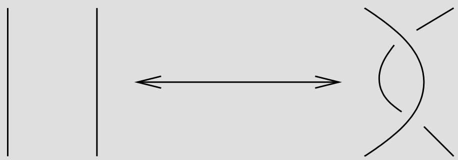

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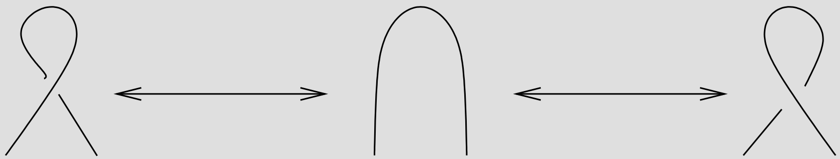
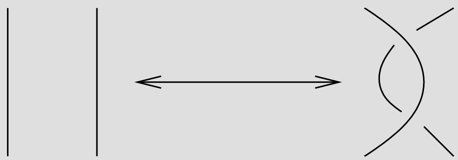
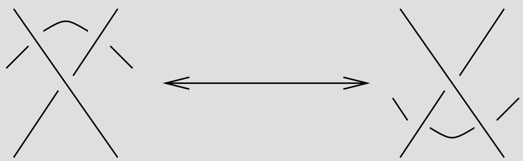
Type I: 

Type II: 

Type III: 

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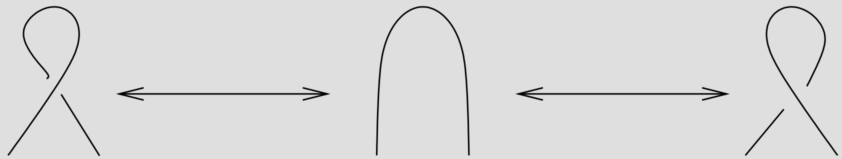
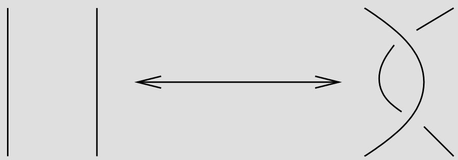
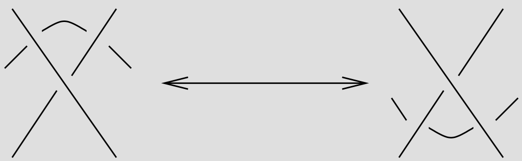
Type I: 

Type II: 

Type III: 

In each of the three types of move, we may replace one picture by the other.

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All known polynomial invariants are not perfect. It may happen that two distinct knots have the same polynomial.

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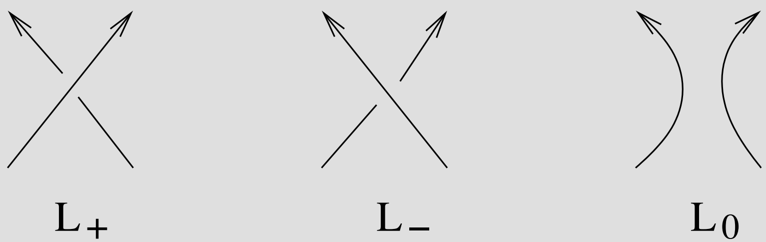
where L_+, L_-, and L_0 are oriented link diagrams that are identical except in a small region where they differ by crossing change or smoothing as in the figure below:



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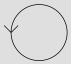
$$V_L /_{\sqrt{t}=-q} = \frac{K(L)}{q + q^{-1}}$$

Computing the Kauffman bracket

Look at Oscar playing with polynomials on the board

Closing remarks

Open problem: Does there exist a knot in $\mathbb{R}^3$, different from the unknot , whose Jones polynomial is 1?

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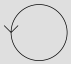
Open problem: Does there exist a knot in $\mathbb{R}^3$, different from the unknot , whose Jones polynomial is 1?

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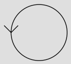
Open problem: Does there exist a knot in $\mathbb{R}^3$, different from the unknot , whose Jones polynomial is 1?

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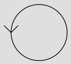
Open problem: Does there exist a knot in $\mathbb{R}^3$, different from the unknot , whose Jones polynomial is 1?

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Recently, researchers are studying ways to generalize the Kauffman bracket in the same sense the Khovanov homology generalizes the Jones polynomial.