j, k-planes of order 4^3

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Abstract

In this article we have constructed and studied a new class of translation planes of order 4^3 that are three dimensional over their kernel. These planes are a generalization of the j-planes discovered by Johnson, Pomareda and Wilke (in 'j-planes'. J. Combin. Theory Ser. A 56 (1991), no. 2, 271–284).

By "defoming" these planes, using (André) net replacement and derivation, it is possible to obtain more new planes of the same order but not necessarily three dimensional over their kernel.

The final count yields thirteen new planes constructed and classified.

The basics

Consider a triple $\Pi = (P, L, \Im)$. Where P is a set of objects that we call points of Π, L is a subset of $\wp(P)$ that we call lines of Π and $\Im$ is a subset of $P \times L$ that we call Incidence.

We will say that a point p is incident with a line l, and that l is incident with p, if $(p, l) \in \Im$. We will also say that p is a point of l and that l contains p.

Two lines, l and m, intersect if there exist a point p that is incident with both. If the lines do not intersect, the lines are said to be parallel. This definition induces naturally an equivalence relation in L.

Affine planes

Definition: $\Pi = (P, L, \Im)$ as above is an affine plane if it satisfies the following conditions:

i. There is a unique line incident with any two given points.

ii. For every line l and every point p not incident with l there is a unique line m that is incident with p and that does not intersect l.

iii. There are four points that are not in the same line.

The last condition rules out degenerate cases, it assures that our plane "looks" like a regular Euclidean plane.

Order of a plane

It can be shown that for any finite affine plane Π there is a number n, called the order of the plane, so that:

The number of points in Π is n^2 and the number of lines is $n^2 + n$. Also, there are n points in each line and $n + 1$ lines per point.

All known finite affine planes have order p^h for p prime and $h \geq 1$.

Translation planes

We are interested in a subclass of affine planes that admit a group acting on the lines that fixes all parallel classes and that acts transitively on the affine points of the plane. These planes are called *translation planes*.

Spreads

Definition: Let V be a $2n$-dimensional vector space over a field K. A spread S of V is a set of n-dimensional subspaces of V that intersect trivially and that partition the space.

The elements of S are called components of the spread S.

Note that the direct sum of any two components of S is equal to V.

Relation between spreads and translation planes

Consider V, a $2n$-dimensional vector space over a field K, and let S be a spread of V. We define $\Pi = (P, L, \Im)$ by:

i. The elements of P are the points (vectors) of V

ii. The elements of L are the components of S and all its (additive) cosets.

iii. The incidence is given by the natural set theoretic inclusion.

Theorem: Π is an Affine Plane. Moreover, the order of Π depends on the number of elements of K. If $K = \mathbb{F}_q$ then, the order of Π is q^n.

A little notation

Let V be a $2n$-dimensional vector space over K.

For a fixed matrix $M \in GL(n, K)$, call ℓ_M, or simply $(y = xM)$, to the n-dimensional subspace $\{(x, y) \in V;\ y = xM\}$.

Example of a spread

Let $p(x) = x^3 - ax^2 - bx - c$ be irreducible in $\mathbb{F}_4[x]$.

It is not hard to see that the set F of all matrices of the form

$$M_{r,s,t} = \begin{bmatrix} r & s & t \\ ct & r+bt & s+at \\ c(s+at) & ct+b(s+at) & r+bt+a(s+at) \end{bmatrix},$$

where $r, s, t \in \mathbb{F}_4$, is a field of order 4^3 contained in $GL(3,4) \cup \{0\}$.

Define $S = \{(y=0)\} \cup \{(y = xM)\ ;\ M \in F\}$.

The fact that F is a field implies that S is a spread. Its associated translation plane has order 4^3.

Towards j,k-planes

Definition: Let j and k be two fixed positive integers and let F be the field of the previous example. Define:

$$G = \left\{ \begin{bmatrix} \Delta_M & 0 \\ 0 & M \end{bmatrix} ; \ M \in F^* \right\}$$

where $\Delta_M = \begin{bmatrix} 1 & & \\ & \partial^{-j} & \\ & & \partial^{-k} \end{bmatrix}$ and $\partial = \det(M)$.

It is easy to see that G is a cyclic group of order $4^3 - 1$.

j,k-planes

Consider the 3-dimensional subspace of V given by the formula $y = x$ and let G be the cyclic group defined above.

In case $S = \{(y = 0)\} \cup O_G(y = x)$ is a spread we will say that its associated plane is a j, k-plane.

We want to study the conditions for these planes to exist, find examples, classify them, etc.

Existence

We have shown that the set S is a spread if and only if

$$\det(\Delta^{-1}M - Id) \neq 0 \;\; for\ every\ M \neq Id\ in\ F.$$

Using this result, among others, we have been able to use the computer to determine when j, k-planes of order 4^3 exist.

More planes

By using (André) net replacement and derivation in j,k-planes and in the replaced planes, we have constructed even more planes from the ones already found.

Roughly, replacement and derivation consist in removing a certain set of lines of the plane and replace them by other suitable set of lines.

Derivation and replacement yield translation planes that might or might not be isomorphic to the original one. Also, not every plane is replaceable or derivable.

Important technique used

One of the most important tools used to determine whether or not two planes are isomorphic is to study their corresponding symmetry groups (collineation groups).

In our case, we used the group G, which acts transitively on the non-zero components of S, the subgroup

$$\Gamma = \left\{ \begin{bmatrix} \Delta_M & 0 \\ 0 & M \end{bmatrix} \in G;\ det(M) = 1 \right\}$$

of G, which fixes pointwise the line $(y = 0)$ and setwise the line $(x = 0)$ (what is called a homology group) and, finally, a second homology group of order $4 - 1$ that fixes pointwise the line $(x = 0)$ and setwise the line $(y = 0)$

Results

We were able to show that a j, k-plane that does not admit a collineation interchanging $(x = 0)$ and $(y = 0)$ is new. Moreover, if such a collineation exists, the j, k-plane is a nearfield plane (in particular, an André plane).

We have also shown that there are only three non-isomorphic j, k-planes of order 4^3. One of them is isomorphic to a known plane (André), the other two are new but one can be constructed from the other via André net replacement.

After studying all three families of planes obtained, we have determined that there are 13 non-isomorphic new planes.

Generalizations

We have generalized the notion of a j, k-plane of order 4^3 to, what we call, $jj.....j$-planes of order q^n for every n and every q, q a power of a prime.

Theorem: There are $jj....j$-planes of order q^n for every $q = p^\alpha$ and n dividing $q - 1$. Moreover, there are new $jj....j$-planes of orders 4^3, 5^4, 7^3, 3^4 and 4^4.

References

[1] Johnson, N. L.; Pomareda, R.; Wilke, F. W. j-planes. J. Combin. Theory Ser. A 56 (1991), no. 2, 271–284.

[2] Johnson, N.L.; Vega, O; Wilke F.W. $j, k-$planes of order 4^3. Innovations in Incidence Geometry. To appear.