Translation planes admitting a linear Abelian group of order $(q+1)^2$.

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The Basics.

Definition

Let V be a 4-dimensional vector space over the finite field $GF(q)$.

The 3-dimensional projective space over $GF(q)$, called $PG(3, q)$, is the geometry obtained by considering

i the 1-dimensional subspaces of V as points of $PG(3, q)$,

ii the 2-dimensional subspaces of V as lines of $PG(3, q)$, and

iii the 3-dimensional subspaces of V as planes of $PG(3, q)$

Definition

A spread of $PG(3, q)$ is a set of lines of $PG(3, q)$ that partition the set of points of $PG(3, q)$.

The Basics.

Definition

A translation plane π of order q^2 can be constructed from a spread $\mathcal{S}$ of $PG(3, q)$ by considering the points of π to be the points of $PG(3, q)$ and the lines of π to be the lines of $\mathcal{S}$ plus all their translates.

Definition

A collineation of a translation planes π is a one-to-one map that preserves the incidence relation.

The Basics.

- The group of all collineations of a translation plane π is the semi-direct product of the translations of V and the collineations of π fixing 0 (called the translation complement).

- If a group of collineations of π consists only of linear transformations of V, then the group is said to be 'linear'.

- If a collineation σ of π fixes every point on a line ℓ and every line through a point P (not on ℓ) then σ is said to be a (P, ℓ)-homology, or a homology with axis ℓ and center P.

Introduction to the problem.

- Given a conic on a plane π of $PG(3, q)$ and a point C not on π, we define a quadratic cone with vertex C as the set consisting of the points on the conic and the points on the lines through C and a point of the conic.

- A flock of a quadratic cone is a partition of the points of a quadratic cone, different from its vertex, into q disjoint conics.

- Translation planes with spreads that can be partitioned into reguli that share exactly one line are connected to flocks of quadratic cones.

- In fact this connection extends to other geometric objects, such as q-clans and generalized quadrangles.

What is known... using groups.

Theorem (Johnson)

Translation planes with spreads in $PG(3, q)$ admitting cyclic affine homology groups of order $q + 1$ are equivalent to flocks of quadratic cones.

In a plane like the one in the previous theorem, one can show that the normalizer of the affine homology group of order $q + 1$ must contain a (collineation) group of order $(q + 1)^2$.

It has been conjectured that all translation planes of order q^2 and spread in $PG(3, q)$ admitting a linear collineation group of order $(q + 1)^2$ must be associated to a flock of a quadratic cone.

The problem.

The problem we studied was a particular case of that conjecture:

Problem (Il problema Abeliano rosso)

Determine the translation planes π of order q^2 with spread in $PG(3, q)$ that admit an Abelian collineation group G of order $(q+1)^2$ in $GL(4, q)$.

What we were able to prove was

Theorem (D-V)

Let π be a translation plane of order q^2 (q an odd prime power) with spread in $PG(3, q)$ admitting a linear Abelian collineation group G of order $(q+1)^2$. Assume that G contains at most three kernel homologies and that $q^2 - 1$ admits a p-primitive divisor, then π is associated to a conical flock plane.

Even case? Future work.

Johnson and Pomareda proved that under the same conditions of the previous theorem, in the case q even, the translation plane admitting the collineation group of order $(q+1)^2$ is André or Desarguesian.

We are currently working on the non-Abelian case. This is, as expected, much harder than the Abelian case. Our results are far from proving the conjecture. Also, in the Abelian case, we suspect that by removing hypotheses we should be able to find counterexamples to the conjecture. We have not found any yet.