			T	H	E			
		M	A	T	H			
						O	F	
	S	U	D	O	K	U		

Oscar Vega. March 25^{th}, 2011.

The Game

A Sudoku board is a 9×9 table filled with nine different symbols, usually $1, 2, \cdots, 9$, that has been partitioned into nine 3×3 blocks such that:

- each symbol appears exactly once in each row,
- each symbol appears exactly once in each column,
- each symbol appears exactly once in each block.

The game consists in completing a sudoku board from the few entries given.

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

The Game

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- each symbol appears exactly once in each column,
- each symbol appears exactly once in each block.

The game consists in completing a sudoku board from the few entries given.

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Sudo What?

- The name is derived from *'suji wa dokushin ni kagiru'*, which is the Japanese for *'the digits must remain single'*.
- It is known that this game (essentially the same) was already known in France in the 1800's.

eut la tête prise et écrasée par les tampons.
La mort a été instantanée.

Léopold Lepers.

DIVERTISSEMENTS QUOTIDIENS

N° 3879 — CARRÉ MAGIQUE DIABOLIQUE
Par M. B. Meyniel

Compléter le carré ci-dessous en employant les neuf premiers nombres chacun neuf fois de manière que les horizontales, les verticales et les deux grandes diagonales donnent toujours à l'addition le même total.

7	8	9	1	2	3	4	5	6
3				4				8
5				9				1
8				3				4
1	2	3	4	5	6	7	8	9
6				7				2
9				1				5
2				6				7
4	5	6	7	8	9	1	2	3

Ce carré devra être diabolique, c'est-à-dire que le carré restera magique si l'on place une ligne horizontale ou une colonne verticale à la suite de toutes les autres.

N° 3865 — MATHÉMATIQUES
Par M. Adolphe R.

Solution

Le marchand a vendu 27,075 vases; le 95e jour, le dernier, il a vendu 567 vases.

Solutions justes

MM. Améthyste; un chercheur; Paul et Jules Duplant; Albert Labarre; L. Grobet; C. Gerbaulet.

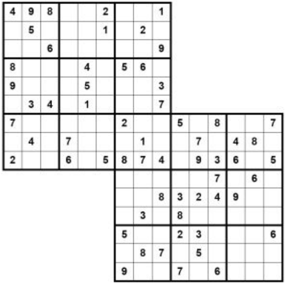
Les solutions et les envois de problèmes inédits doivent être adressés, dans la huitaine, au rédacteur soussigné.

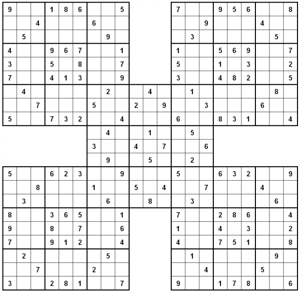
Félix André.

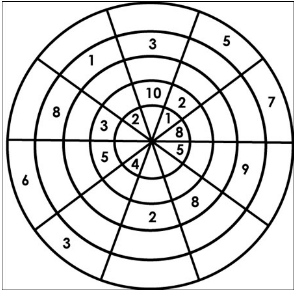
Sudo-History

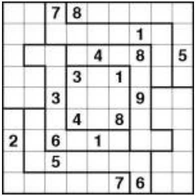
- The modern version of Sudoku is an American invention (not Japanese). The author is unknown, but there is evidence it was Howard Garns.
- First published by Dell Puzzle Magazine in 1979 under the name *Number Place*.
- Taken by the Japanese puzzle company Nikoli in the mid-80's. That is when the name Sudoku was originated.
- In 2005, Felgenhauer and Jarvis proved that there are 6,670,903,752,021,072,936,960 Sudoku boards. Russell and Jarvis proved that there are 5,472,730,538 *essentially different* boards.

Pop Culture: Cool, Awesome... X-treme!!!

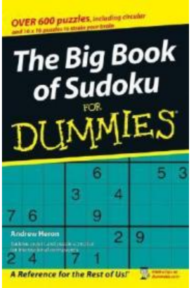


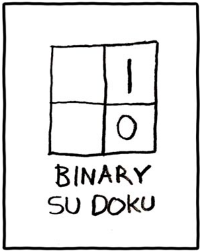






Pop Culture: Silly, Funny,.... WTH?





Math Game?

People say that sudoku is a math game because numbers are used to play.

But any set of nine distinct symbols can be used to play.

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

$\longrightarrow$

E	C			G				
F			A	I	E	C		
	I	H					F	
H				F				C
D			H		C			A
G				D				F
	F					B	H	
			D	A	I			E
				H			G	I

Math Game?

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But any set of nine distinct symbols can be used to play.

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

$\longrightarrow$

$\diamondsuit$	$\aleph$			$\spadesuit$				
$\clubsuit$			$\int$	∞	$\diamondsuit$	$\aleph$		
	∞	$\otimes$					$\clubsuit$	
H				$\clubsuit$				$\aleph$
∂			$\otimes$		$\aleph$			$\int$
$\spadesuit$				∂				$\clubsuit$
	$\clubsuit$					$\heartsuit$	$\otimes$	
			∂	$\int$	∞			$\diamondsuit$
				$\otimes$			$\spadesuit$	∞

Having to use numbers to fill the cells does not make it a math game.
Then, what does?

Logic Game

- In order to succeed solving Sudoku puzzles one has to use logic!
- Can the techniques used to play sudoku be seen as techniques used to proof mathematical theorems?

I might be pushing it here, let us see a few examples.

Direct Proof

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

$\longrightarrow$

5	3			7				
6			1	9	5	3		
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
	8		4	1	9			5
				8			7	9

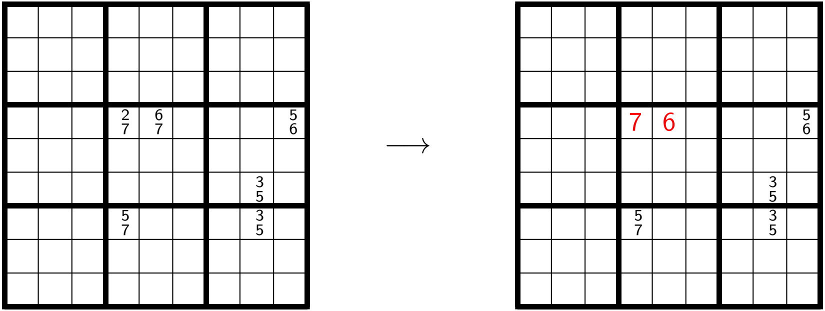
Forcing Chains (Proof by Contradiction)

			2 7	6 7				5 6
							3 5	
			5 7				3 5	

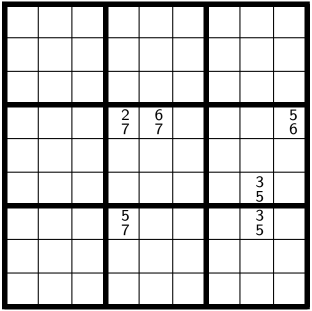
⟶

			7	6 7				5 6
							3 5	
			5 7				3 5	

Forcing Chains (Proof by Contradiction)



Forcing Chains (Proof by Contradiction)



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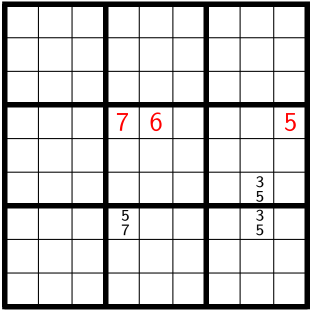

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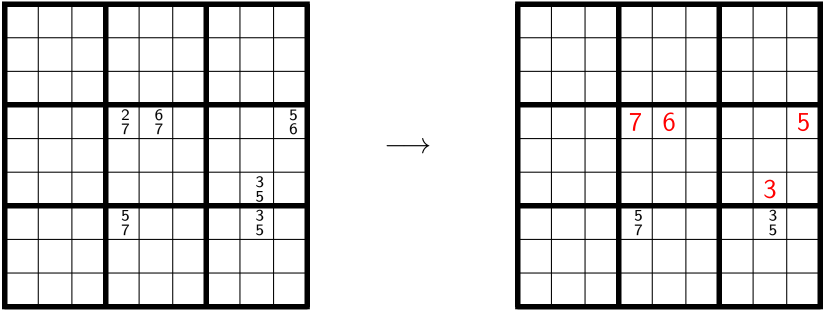

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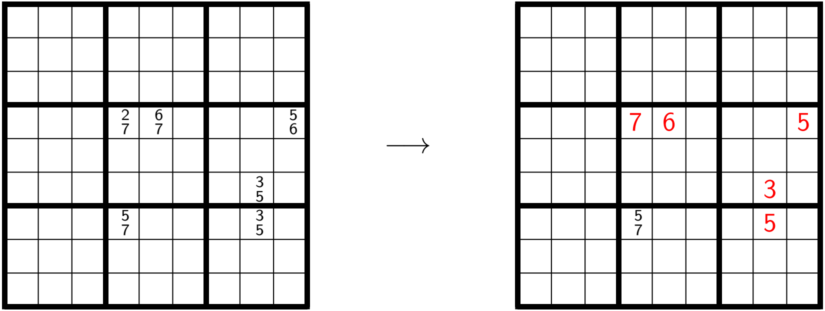
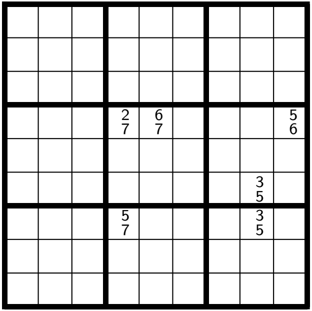
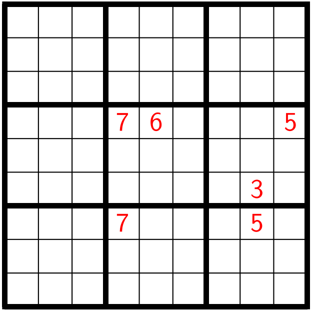


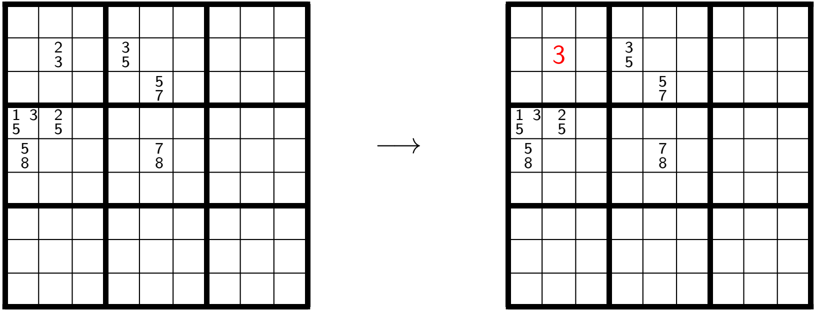
Forcing Chains (Proof by Contradiction)



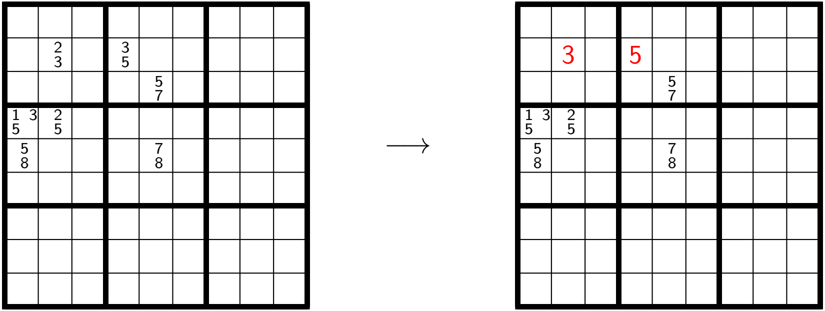


CONTRADICTION!!!

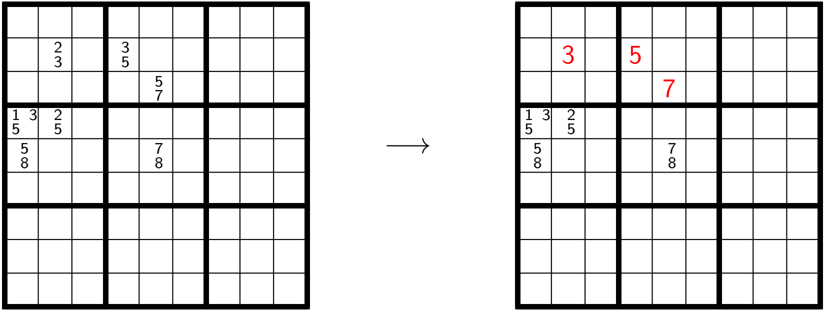
Forcing Chains (Proof by Cases)



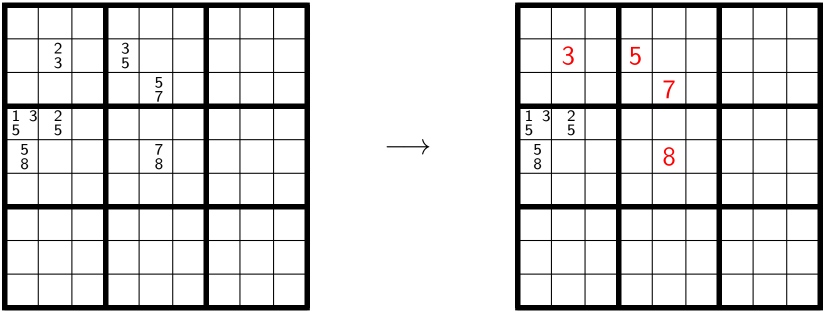
Forcing Chains (Proof by Cases)



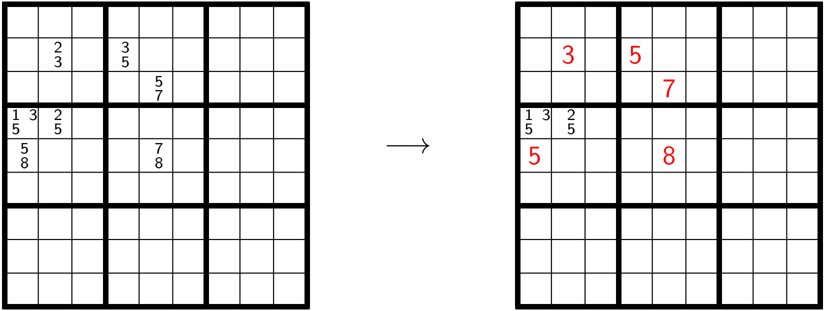
Forcing Chains (Proof by Cases)



Forcing Chains (Proof by Cases)

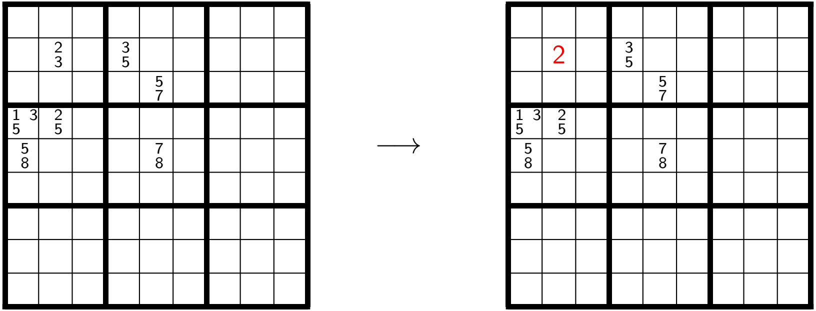


Forcing Chains (Proof by Cases)

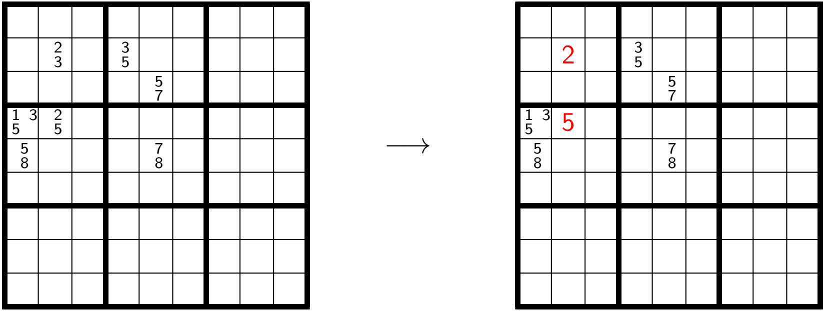


So, 5 cannot be in the square with options $1, 3, 5$ (A4).

Forcing Chains (Proof by Cases)



Forcing Chains (Proof by Cases)



So, 5 cannot be in the square with options $1, 3, 5$ (A4). **In either case!!**

Many-to-one

8	3	5	4	1	6	9	2	7
2	9	6	8	5	7	4	3	1
4	1	7	2	9	3	6	5	8
5	6	9	1	3	4	7	8	2
1	2	3	6	7	8	5	4	9
7	4	8	5	2	9	1	6	3
6	5	2	7	8	1	3	9	4
9	8	1	3	4	5	2	7	6
3	7	4	9	6	2	8	1	5

We can create many other sudoku boards out of this one by:

- rotating it,
- interchanging ‘stacks’,
- interchanging ‘bands’,
- interchanging columns in a stack,
- interchanging rows in a band,
- relabeling of the digits.

Isomorphisms

Boards that can be connected to each other using (finitely many of) the tricks described in the previous slide are said to be *essentially the same*. How many boards can we get out of a single given board?

- Starting with one Sudoku puzzle, a daily calendar of Sudoku puzzles can be produced... enough for the entire next century! [1].
- There are 6,670,903,752,021,072,936,960 distinct Sudoku boards [2].
- There are 5,472,730,538 *essentially different* boards [3].

---

[1] Bartlett & Langville. An Integer Programming Model for the Sudoku Model. The Journal of Online Mathematics and Its Applications, Vol. 8 (2008)

[2] Felgenhauer & Jarvis, Mathematics of Sudoku I (2006)

[3] Russel & Jarvis. Mathematics of Sudoku II (2006)

Minimality

Q. What is the minimum number of 'givens' in a Sudoku board that are needed to have unique solution?

A. Unknown.
The minimum number known is 17. Gordon Royle has a collection of 49151 distinct Sudoku configurations with 17 entries at

http://mapleta.maths.uwa.edu.au/∼gordon/sudokumin.php

- Gary McGuire is conducting a search for a 16-given Sudoku board with unique solution. He developed a program, called CHECKER for this purpose. It seems that the program requires about 300,000 years (on a one-core computer equipped with CPU, Intel(R) Xeon(R) E5520 @ 2.27GHz) to finish the job.

http://www.math.ie/checker.html

Uniqueness

A well-defined Sudoku board must have a unique solution. This can be used to solve Sudoku boards.

Hence, the following situation is impossible!!

5	1	6	8	2	7	9	4	3
2	7	8	3	9	4	6	1	5
3	4	9	6	1	5	8	7	2
9	8	3 7	4	3 7	2	1	5	6
4	6	5	1	8	9	2	3	7
1	2	3 7	5	3 7	6	4	8	9
8	9	2	7	4	3	5	6	1
6	3	4	9	5	1	7	2	8
7	5	1	2	6	8	3	9	4

The graph of a Sudoku board

Herzberg and Ram Murty associated the following graph to a Sudoku board.

- vertices = cells.
- edge between two vertices = two cells are ‘buddies’.

				B				
				B				
				B				
B	B	B	B	C	B	B	B	B
			B	B	B			
			B	B	B			
				B				
				B				
				B				

Coloring a Sudoku board

Herzberg and Ram Murty proved[4] that the coloring of the graph described in the previous slide is related to the uniqueness of solutions for Sudoku boards.

The coloring of the vertices is obvious: The nine digits used represent 9 distinct colors.

Note that solving a Sudoku puzzle is nothing but (the mathematical task of) extending a partial vertex coloring to a valid 9-coloring of the entire graph.

Theorem (Herzberg & Ram Murty)

The chromatic number of a Sudoku puzzle is 9.

[4] Sudoku Squares and Chromatic Polynomials. Notices of the AMS. Vol. 54, Number 6 (2007)

More theorems

Theorem (Herzberg & Ram Murty)

Let G be a graph with chromatic number $C(G)$ and P be a partial coloring of G using only $C(G) - 2$ colors. If P can be completed to a total proper coloring of G, then there are at least two ways of extending the coloring.

Corollary

Let n be distinct numbers in a given Sudoku puzzle. If $n \leq 7$ then the solution of the given Sudoku puzzle is not unique.

In a related note: the following theorem is due to Sander[5]

Theorem

The eigenvalues of the adjacency matrix of the Sudoku graph are integers.

---

[5] Sudoku graphs are integral. The electronic journal of combinatorics 16 (2009), #N25

Latin squares

- Introduced and studied by Leonhard Euler in the 1700's.
- A Latin square of order n is an $n \times n$ table filled with n different symbols in such a way that:

  each symbol appears exactly once in each row, and each column.

- Examples: a Latin square of order 4 with entries $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ and one of order 5 with entries {a, b, c, d, e} :

$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$
$\heartsuit$	$\diamondsuit$	$\clubsuit$	$\spadesuit$
$\diamondsuit$	$\clubsuit$	$\spadesuit$	$\heartsuit$
$\clubsuit$	$\spadesuit$	$\heartsuit$	$\diamondsuit$

a	b	c	d	e
b	c	e	a	d
c	e	d	b	a
d	a	b	e	c
e	d	a	c	b

Latin squares of order 9

Let us look at a Latin square of order 9 with entries {1, 2, 3, 4, 5, 6, 7, 8, 9} :

8	3	2	5	9	1	6	7	4
1	8	5	7	4	6	3	9	2
5	7	1	2	6	4	9	8	3
4	9	6	3	8	7	2	5	1
2	6	7	9	5	3	4	1	8
9	4	3	8	1	2	7	6	5
7	1	4	6	3	8	5	2	9
3	2	9	1	7	5	8	4	6
6	5	8	4	2	9	1	3	7

Reminded you of something?

Gerechte designs

- Introduced by the German statistician W.U. Behrens in 1956.
- A gerechte design is an $n \times n$ table filled with n different symbols and partitioned into n regions, each containing n small squares of the grid, in such a way that :

  each symbol occurs exactly once in each row, each column,and each region.
- Example: two gerechte designs of order 9:

6	9	5	3	4	2	8	1	7
8	7	9	6	3	1	5	4	2
1	5	2	8	6	9	4	7	3
3	1	4	5	2	7	6	9	8
5	3	7	4	9	6	2	8	1
9	4	6	1	8	3	7	2	5
7	2	8	9	5	4	1	3	6
4	8	1	2	7	5	3	6	9
2	6	3	7	1	8	9	5	4

8	3	5	4	1	6	9	2	7
2	9	6	8	5	7	4	3	1
4	1	7	2	9	3	6	5	8
5	6	9	1	3	4	7	8	2
1	2	3	6	7	8	5	4	9
7	4	8	5	2	9	1	6	3
6	5	2	7	8	1	3	9	4
9	8	1	3	4	5	2	7	6
3	7	4	9	6	2	8	1	5

THANK YOU