Symplectic Planes

Discrete Mathematics Seminar, UCDHSC-DDC. February 19, 2007

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$*$ We will say that $\mathcal{B}$ is non-degenerate if B has rank 4.

A nice representation for $\mathcal{B}$

$*$ It is known that all non-degenerate symplectic forms of V are equivalent. Moreover, it is possible to find a basis of V such that

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$*$ This basis is called a symplectic basis.

Totally isotropic subspaces of V relative to $\mathcal{B}$

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$*$ The maximal totally isotropic subspaces of V have dimension 2.

$*$ The geometry formed by calling 'points' to the one-dimensional subspaces of V and 'lines' to all the 2-dimensional totally isotropic subspaces of V is called $\mathcal{W}(q)$. It is known that $\mathcal{W}(q)$ is a GQ of order (q, q).

An example

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So, S is a totally isotropic subspace of V relative to $\mathcal{B}$.

Spreads of V

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$*$ An affine plane Π may be constructed from a spread S. The points of Π are the vectors of V, the lines of Π are the components of S together with all their additive cosets. The incidence is the obvious one.

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$*$ For every 2-dimensional subspace S of V that intersects $(x=0)$ trivially there is a matrix $M\in M_2(K)$ such that

$$S=\{(x,y,a,b);\ (a,b)=(x,y)M\}$$

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$\ast$ It follows that every spread of V containing $(x = 0)$ can be considered as a subset of $M_2(K)$.

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$*$ Any such a spread will be called a symplectic spread, and the plane constructed from the spread will be called a symplectic plane.

Symplectic spreads = symmetric matrices

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$*$ So, $M = M^t$ for all matrices of a symplectic spread.

More on spreads

$*$ Assuming that $(x = 0)$ is one of the components of a spread S, then the other q^2 matrices of $S = \{M_i;\ i = 1, 2, ..., q^2\}$ satisfy

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$*$ Using this condition we can show that the set

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is a spread iff for every $a \in GF(q)$ the function

$$(t, u) \mapsto (at + f(t,u), au + g(t,u))$$

is a permutation of $GF(q) \times GF(q)$.

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is a symplectic spread of V.

Known symplectic spreads of V

$Name$	$g(x,y)$	q	$Restrictions$
Regular	$-nx$	odd	n non-square
Kantor	$-nx^{\alpha}$	odd	n non-square, $\alpha \mid q$
Thas-Payne	$-nx-(n^{-1}x)^{1/9}-y^{1/3}$	3^h	n non-square, $h>2$
Penttila-Williams	$-x^9-y^{81}$	3^5	
Ree-Tits slice	$-x^{2\alpha+3}-y^{\alpha}$	3^{2h+1}	$\alpha=\sqrt{3q}$
Regular	$cx+y$	even	$Tr_{q\to 2}(c)=1$
Tits-Lüneburg	$x^{\alpha+1}+y^{\alpha}$	2^{2h+1}	$\alpha=\sqrt{2q}$

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$*$ Actually, most of the symplectic planes in the previous list have been found by people studying ovoids, not planes.

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∗ Norm and Stan are responsible of me getting into the first problem, and Bill into the second.