Symplectic Translation Planes

Rocky Mountain Algebraic Combinatorics Seminar, CSU.

February 23, 2007.

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$*$ The maximal totally isotropic subspaces of V have dimension n (Witt index $= n$).

n-dimensional subspaces of V

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$\ast$ Note that $(x = 0) = \{(x, y) \in V;\ x = 0\}$ is an n-dimensional subspace of V.

$\ast$ Every n-dimensional subspace S of V that intersects $(x = 0)$ trivially can be represented as

$$(y = xM) = \{(x, y);\ y = xM\}$$

for some $M \in M_n(K)$.

Maximal totally isotropic subspaces of V

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for all x, a, which forces $M = M^t$.

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$\ast$ If S is a spread having all components totally isotropic, then $M = M^t$ for all matrices of S (in some basis).

Symplectic translation planes

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$*$ A spread having only totally isotropic components will be said to be a symplectic spread. The associated translation plane is called a symplectic plane.

Kerdock sets

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$*$ Moreover, it is possible to choose a basis of V such that besides the spread being symmetric, the zero matrix is also in the spread.

$*$ Thus, for q even, the spreadset of Π is a Kerdock set. These sets are related to Kerdock codes.

A couple of results by Kantor

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$*$ **Theorem:** Let Σ be a symplectic spread of the F vector space V. Let K be the kernel of Σ. Assume that $g \in \Gamma L(V)$ is a collineation of Σ and that either $|F|$ is even or $[K : F]$ is odd, then $g = sh$, where $h \in K$ and $s \in \Gamma Sp(V)$.

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$\ast$ **Theorem:** A symplectic plane may be considered as having a symplectic spread over its kernel. That is, the symplectic dimension of a plane is equal to the dimension of a plane over its kernel.

More on symplectic spreads

$*$ The set

$$S = \left\{ \begin{bmatrix} t & u \\ u & g(t,u) \end{bmatrix} ;\ t, u \in GF(q) \right\}$$

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is a permutation of $GF(q)$.

Known symplectic spreads of $PG(3,q)$

$Name$	$g(x,y)$	q	$Restrictions$
Regular	$-nx$	odd	n non-square
Kantor	$-nx^{\alpha}$	odd	n non-square, $\alpha \vert q$
Thas-Payne	$-nx-(n^{-1}x)^{1/9}-y^{1/3}$	3^h	n non-square, $h>2$
Penttila-Williams	$-x^9-y^{81}$	3^5	
Ree-Tits slice	$-x^{2\alpha+3}-y^{\alpha}$	3^{2h+1}	$\alpha=\sqrt{3q}$
Regular	$cx+y$	even	$Tr_{q\to 2}(c)=1$
Lüneburg	$x^{\alpha+1}+y^{\alpha}$	2^{2h+1}	$\alpha=\sqrt{2q}$

Symplectic planes of even order and ovoids of $PG(3, q)$

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$\ast$ The dual of a symplectic spread is a set of q^2+1 points in $\mathcal{W}(q)$ so that no three of them are collinear. That is, they form an ovoid of $\mathcal{W}(q)$... which ends up being an ovoid of $PG(3,q)$.

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$\ast$ The dual of a symplectic spread is a set of $q^2 + 1$ points in $\mathcal{W}(q)$ so that no three of them are collinear. That is, they form an ovoid of $\mathcal{W}(q)$... which ends up being an ovoid of $PG(3, q)$.

$\ast$ We can say that, for q even, the study of ovoids of $PG(3, q)$ is equivalent to the study of symplectic planes of order q^2 and kernel containing $GF(q)$.

Symplectic planes and corresponding ovoids

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$Plane$	$g(x,y)$	q	$Restrictions$	$Ovoids$
Desarguesian	$cx+y$	even	$Tr_{q\to 2}(c)=1$	Elliptic quadric
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$*$ For q odd all ovoids of $PG(3,q)$ are elliptic quadrics. On the other hand, we have seen that there are several symplectic planes that are non-isomorphic.

Interesting (solved) problem

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$*$ **Theorem:** An ovoid of $PG(3, q)$, q even, having a bundle of conics through a point must be an elliptic quadric.

$*$ **Theorem:** A symplectic conical flock plane of order q^2 with kernel containing $GF(q)$ must be Desarguesian or Kantor.

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$*$ **I believe** this result can be generalized to q odd by allowing the plane to be a Kantor plane.

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$\ast$ **Theorem:** A conical symplectic plane must be Desarguesian or Kantor.

I have been able to prove (so far)

Theorem: Let $(V, \mathcal{B})$ be a symplectic space of dimension 4, and let Π be a symplectic plane with spread S of V. Assume the order of the plane is even and that it contains a regulus R. Then Π is covered by nets of k Desarguesian planes such that

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$*$ All Desarguesian planes of the covering are symplectic with respect to $\mathcal{B}$.

$*$ The planes of the covering pairwise share R (and nothing else).

$*$ Any collineation of Π that stabilizes R is a collineation of every Desarguesian plane of the covering.

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Remark: The covering described above defines a Desarguesian decomposition of Π.

More (even weaker) results

$*$ Let Π be a symplectic plane of odd order that contains a derivable net R of size $q+1$. Then Π is covered by Kantor planes that pairwise share R, and nothing else. Moreover, all these Kantor planes are symplectic with respect to $\mathcal{B}$.

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$\ast$ I expect to prove that for even order a similar result holds, but with Lüneburg planes instead of Kantor planes.

$\ast$ Note that all these results for even-order planes mean partitions of ovoids by using elliptic quadrics or, in case the previous paragraph turns out being true, Tits ovoids.

Closing remarks

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∗ I guess that is material for another talk some other time.