The well-covered dimension of a graph

Oscar Vega
California State University, Fresno
(joint work with I. Birnbaum)

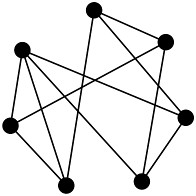
Mathematics Department Seminar. CSU, Fresno. March 10^{th}, 2010.

Definitions.

- A graph is a pair $G = (V, E)$, where V (or $V(G)$) is a set of vertices and E (or $E(G)$) a set edges. Edges can be thought of as segments that contain exactly two vertices.

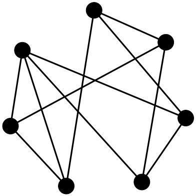
Definitions.

- A graph is a pair $G = (V, E)$, where V (or $V(G)$) is a set of vertices and E (or $E(G)$) a set edges. Edges can be thought of as segments that contain exactly two vertices.



Definitions.

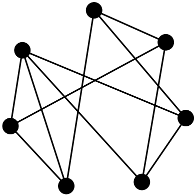
- A graph is a pair $G = (V, E)$, where V (or $V(G)$) is a set of vertices and E (or $E(G)$) a set edges. Edges can be thought of as segments that contain exactly two vertices.



- Vertices that are joined by an edge are called adjacent.

Definitions.

- A graph is a pair $G = (V, E)$, where V (or $V(G)$) is a set of vertices and E (or $E(G)$) a set edges. Edges can be thought of as segments that contain exactly two vertices.



- Vertices that are joined by an edge are called adjacent.

- We will assume there is at most one edge joining two vertices, and that a vertex cannot be adjacent to itself. i.e. No multigraphs and no loops.

Maximal independent sets.

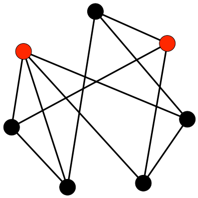
- An independent set (of vertices) is a set of vertices that are pairwise nonadjacent.

Maximal independent sets.

- An independent set (of vertices) is a set of vertices that are pairwise nonadjacent.
- M is a maximal independent set iff M is an independent set such that there is no independent set containing it properly.

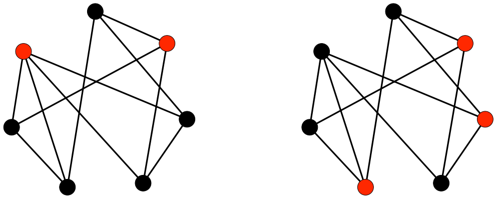
Maximal independent sets.

- An independent set (of vertices) is a set of vertices that are pairwise nonadjacent.
- M is a maximal independent set iff M is an independent set such that there is no independent set containing it properly.



Maximal independent sets.

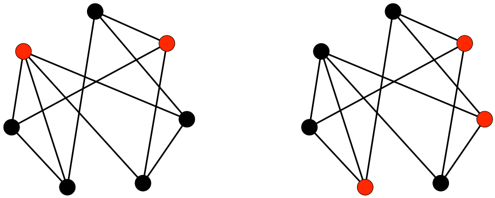
- An independent set (of vertices) is a set of vertices that are pairwise nonadjacent.
- M is a maximal independent set iff M is an independent set such that there is no independent set containing it properly.



Maximal independent sets.

- An independent set (of vertices) is a set of vertices that are pairwise nonadjacent.

- M is a maximal independent set iff M is an independent set such that there is no independent set containing it properly.



- A graph such that every maximal independent has the same cardinality is said to be a well-covered graph.

Weight functions.

Let G be a graph and $\mathbb{F}$ a field.

Weight functions.

Let G be a graph and $\mathbb{F}$ a field.

- A function $w : V(G) \longrightarrow \mathbb{F}$ is called a weight function of G.

Weight functions.

Let G be a graph and $\mathbb{F}$ a field.

- A function $w : V(G) \to \mathbb{F}$ is called a weight function of G.
- A weight function w of G such that $\sum_{v \in M} w(v)$ takes the same value for all maximal independent sets M of G is called a well-covered weighting of G.

The well-covered dimension of a graph.

- The set of all well-covered weightings of a graph G is an $\mathbb{F}$-vector space, called the well-covered space of G.

The well-covered dimension of a graph.

- The set of all well-covered weightings of a graph G is an $\mathbb{F}$-vector space, called the well-covered space of G. The dimension of this space is called the well-covered dimension of G, it is denoted $wcdim(G, \mathbb{F})$.

The well-covered dimension of a graph.

- The set of all well-covered weightings of a graph G is an $\mathbb{F}$-vector space, called the well-covered space of G. The dimension of this space is called the well-covered dimension of G, it is denoted $wcdim(G, \mathbb{F})$.

- Assume the maximal independent sets of G are $M_1, \cdots, M_k$.

The well-covered dimension of a graph.

- The set of all well-covered weightings of a graph G is an $\mathbb{F}$-vector space, called the well-covered space of G. The dimension of this space is called the well-covered dimension of G, it is denoted $wcdim(G, \mathbb{F})$.

- Assume the maximal independent sets of G are $M_1, \cdots, M_k$.

  A well-covered weighting w of G is determined by a solution of the homogeneous system of linear equations formed by selecting, let us say M_1, and setting the system $M_i - M_1 = 0$ for $i = 2, \cdots, k$.

The well-covered dimension of a graph.

- The set of all well-covered weightings of a graph G is an $\mathbb{F}$-vector space, called the well-covered space of G. The dimension of this space is called the well-covered dimension of G, it is denoted $wcdim(G, \mathbb{F})$.

- Assume the maximal independent sets of G are $M_1, \cdots, M_k$.

  A well-covered weighting w of G is determined by a solution of the homogeneous system of linear equations formed by selecting, let us say M_1, and setting the system $M_i - M_1 = 0$ for $i = 2, \cdots, k$.

  If A is the matrix associated to this system then

$$wcdim(G, \mathbb{F}) = |V(G)| - rank(A)$$

Basic results.

- (Brown & Nowakowski) Let $G = C_1 \cup \cdots \cup C_k$, be a graph partitioned into k connected components. Then, for all fields $\mathbb{F}$,

$$wcdim(G, \mathbb{F}) = \sum_{i=1}^{k} wcdim(C_i)$$

Basic results.

- (Brown & Nowakowski) Let $G = C_1 \cup \cdots \cup C_k$, be a graph partitioned into k connected components. Then, for all fields $\mathbb{F}$,

$$wcdim(G, \mathbb{F}) = \sum_{i=1}^{k} wcdim(C_i)$$

- $wcdim(\overline{K_n}, \mathbb{F}) = n$, for all $n \in \mathbb{N}$ and all fields $\mathbb{F}$.

Basic results.

- (Brown & Nowakowski) Let $G = C_1 \cup \cdots \cup C_k$, be a graph partitioned into k connected components. Then, for all fields $\mathbb{F}$,

$$wcdim(G, \mathbb{F}) = \sum_{i=1}^{k} wcdim(C_i)$$

- $wcdim(\overline{K_n}, \mathbb{F}) = n$, for all $n \in \mathbb{N}$ and all fields $\mathbb{F}$.

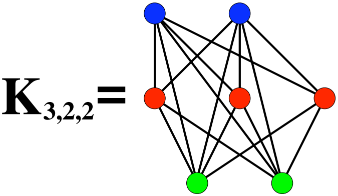
- $wcdim(K_n, \mathbb{F}) = 1$, for all $n \in \mathbb{N}$ and all fields $\mathbb{F}$.

Not so basic results.

- (B & V) Complete multipartite graphs:

Not so basic results.

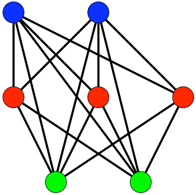
- (B & V) Complete multipartite graphs:



Not so basic results.

- (B & V) Complete multipartite graphs:

$\mathbf{K_{3,2,2}}$ =

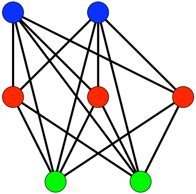


$$wcdim(K_{n_1,\ldots,n_k}, \mathbb{F}) = \sum_{i=1}^{k} n_i - (k-1)$$

for all fields $\mathbb{F}$.

Not so basic results.

- (B & V) Complete multipartite graphs:

$\mathbf{K_{3,2,2}=}$ 

$$wcdim(K_{n_1,\ldots,n_k}, \mathbb{F}) = \sum_{i=1}^{k} n_i - (k-1)$$

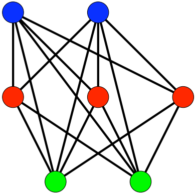
for all fields $\mathbb{F}$.

- Turán graph $T(n, r)$ (K_{r+1}-free graph with the greatest number of edges on n vertices). Then for all fields $\mathbb{F}$,

Not so basic results.

- (B & V) Complete multipartite graphs:

$\mathbf{K_{3,2,2}} =$



$$wcdim(K_{n_1,\ldots,n_k}, \mathbb{F}) = \sum_{i=1}^{k} n_i - (k-1)$$

  for all fields $\mathbb{F}$.

- Turán graph $T(n,r)$ (K_{r+1}-free graph with the greatest number of edges on n vertices). Then for all fields $\mathbb{F}$,

  $$wcdim(T(n,r), \mathbb{F}) = (n \bmod r)\lceil n/r \rceil + (r - (n \bmod r))\lfloor n/r \rfloor - (r-1)$$

Crown graphs.

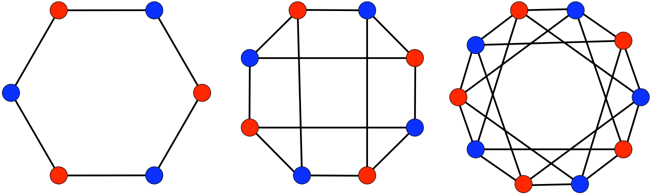
- For $n > 2$, the crown graph S_n^0 is constructed by removing a perfect matching from $K_{n,n}$.

Crown graphs.

- For $n > 2$, the crown graph S_n^0 is constructed by removing a perfect matching from $K_{n,n}$. That is, S_n^0 is $K_{n,n}$ with exactly one edge less per vertex.

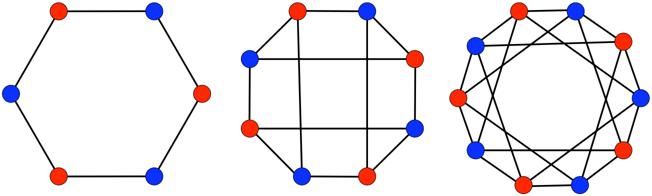
Crown graphs.

- For $n > 2$, the crown graph S_n^0 is constructed by removing a perfect matching from $K_{n,n}$. That is, S_n^0 is $K_{n,n}$ with exactly one edge less per vertex.



Crown graphs.

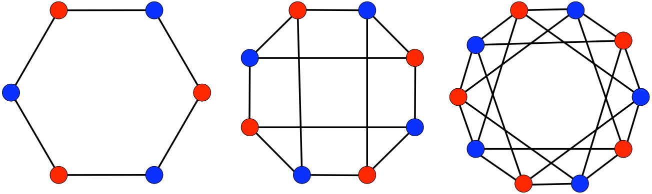
- For $n > 2$, the crown graph S_n^0 is constructed by removing a perfect matching from $K_{n,n}$. That is, S_n^0 is $K_{n,n}$ with exactly one edge less per vertex.



- Brown and Nowakowski found, using methods different from the ones used by us, the well-covered dimension of crown graphs over a field with characteristic 0 or 2.

Crown graphs.

- For $n > 2$, the crown graph S_n^0 is constructed by removing a perfect matching from $K_{n,n}$. That is, S_n^0 is $K_{n,n}$ with exactly one edge less per vertex.



- Brown and Nowakowski found, using methods different from the ones used by us, the well-covered dimension of crown graphs over a field with characteristic 0 or 2.

- (B & V) Let $\mathbb{F}_p$ be a field with characteristic p (p could be zero).

$$wcdim(S_n^0, \mathbb{F}_p) = \begin{cases} n & \text{if } p|(n-2) \\ n-1 & \text{otherwise} \end{cases}$$

Paths and cycles.

- The behavior of paths and cycles is very similar. In fact, we used results on paths to find the well-covered dimension of all cycles.

Paths and cycles.

- The behavior of paths and cycles is very similar. In fact, we used results on paths to find the well-covered dimension of all cycles.

- (B & V) If w is a well-covered weighting of P_n and $n \geq 5$, then

$$w(v_1) = w(v_2), \quad w(v_3) = \ldots = w(v_{n-2}) = 0, \quad w(v_{n-1}) = w(v_n)$$

Paths and cycles.

- The behavior of paths and cycles is very similar. In fact, we used results on paths to find the well-covered dimension of all cycles.

- (B & V) If w is a well-covered weighting of P_n and $n \geq 5$, then

$$w(v_1) = w(v_2), \quad w(v_3) = \ldots = w(v_{n-2}) = 0, \quad w(v_{n-1}) = w(v_n)$$

  Moreover, $wcdim(P_2, \mathbb{F}) = 1$ and $wcdim(P_n, \mathbb{F}) = 2$ if $n > 1$, for all fields $\mathbb{F}$.

Paths and cycles.

- The behavior of paths and cycles is very similar. In fact, we used results on paths to find the well-covered dimension of all cycles.

- (B & V) If w is a well-covered weighting of P_n and $n \geq 5$, then

$$w(v_1) = w(v_2), \quad w(v_3) = \ldots = w(v_{n-2}) = 0, \quad w(v_{n-1}) = w(v_n)$$

  Moreover, $wcdim(P_2, \mathbb{F}) = 1$ and $wcdim(P_n, \mathbb{F}) = 2$ if $n > 1$, for all fields $\mathbb{F}$.

- (B & V) For all fields $\mathbb{F}$.
  (i) $wcdim(C_n, \mathbb{F}) = 0$, if $n \geq 8$,
  (ii) $wcdim(C_n, \mathbb{F}) = 1$, if $n = 3, 5, 7$,
  (iii) $wcdim(C_6, \mathbb{F}) = 2$,
  (iv) $wcdim(C_4, \mathbb{F}) = 3$.

Paths and cycles.

- The behavior of paths and cycles is very similar. In fact, we used results on paths to find the well-covered dimension of all cycles.

- (B & V) If w is a well-covered weighting of P_n and $n \geq 5$, then

  $$w(v_1) = w(v_2), \quad w(v_3) = \ldots = w(v_{n-2}) = 0, \quad w(v_{n-1}) = w(v_n)$$

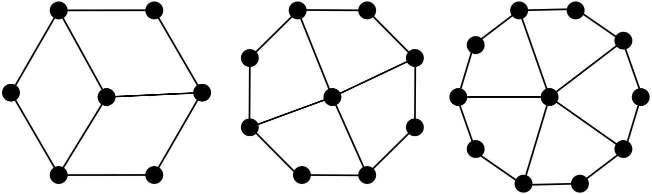
  Moreover, $wcdim(P_2, \mathbb{F}) = 1$ and $wcdim(P_n, \mathbb{F}) = 2$ if $n > 1$, for all fields $\mathbb{F}$.

- (B & V) For all fields $\mathbb{F}$.
  (i) $wcdim(C_n, \mathbb{F}) = 0$, if $n \geq 8$,
  (ii) $wcdim(C_n, \mathbb{F}) = 1$, if $n = 3, 5, 7$,
  (iii) $wcdim(C_6, \mathbb{F}) = 2$,
  (iv) $wcdim(C_4, \mathbb{F}) = 3$.

- The well-covered dimension of paths had been already computed by Caro & Yuster using methods different from the ones we used.

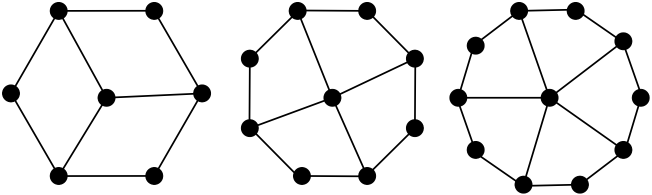
Gear graphs.

- Now we look at the family of gear graphs. These graphs are given by



Gear graphs.

- Now we look at the family of gear graphs. These graphs are given by



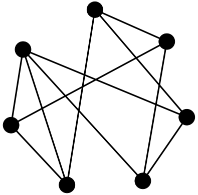
- (B & V) Let G_n be the gear graph in $2n+1$ vertices, then

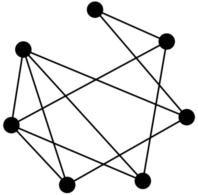
$$wcdim(G_n, \mathbb{F}) = \begin{cases} 3 & \text{if } n = 3 \\ 0 & \text{if } n > 3 \end{cases}$$

for all fields $\mathbb{F}$.

Two special graphs.

- (B & V) The graph SG_1 (left picture below)

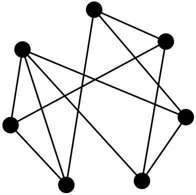


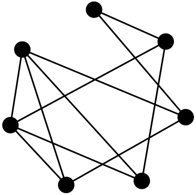


is the 'smallest' graph known to have well-covered dimension depending on the characteristic of $\mathbb{F}$.

Two special graphs.

- (B & V) The graph SG_1 (left picture below)



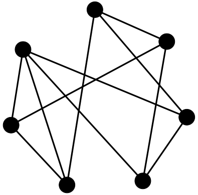


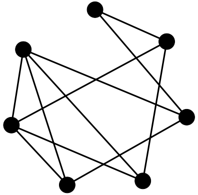
is the 'smallest' graph known to have well-covered dimension depending on the characteristic of $\mathbb{F}$. In fact, if $\mathbb{F}_p$ is a field with characteristic p (p could be zero), then

$$wcdim(SG_1, \mathbb{F}_p) = \begin{cases} 3 & \text{if } p = 2 \\ 2 & \text{otherwise} \end{cases}$$

Two special graphs.

- (B & V) The graph SG_1 (left picture below)





is the 'smallest' graph known to have well-covered dimension depending on the characteristic of $\mathbb{F}$. In fact, if $\mathbb{F}_p$ is a field with characteristic p (p could be zero), then

$$wcdim(SG_1, \mathbb{F}_p) = \begin{cases} 3 & \text{if } p = 2 \\ 2 & \text{otherwise} \end{cases}$$

- (B & V) The graph SG_2 (right picture above) is the 'smallest' graph known to have well-covered dimension equal to zero for all fields $\mathbb{F}$.

Blowups of graphs I

- Let G be a graph and $t \in \mathbb{N}$. A t-blowup of a vertex $v_i \in V(G)$ is an independent set $V_{v_i} = \{v_{i1}, v_{i2}, \cdots, v_{it}\}$ that 'takes the place' of v_i.

Blowups of graphs I

- Let G be a graph and $t \in \mathbb{N}$. A t-blowup of a vertex $v_i \in V(G)$ is an independent set $V_{v_i} = \{v_{i1}, v_{i2}, \cdots, v_{it}\}$ that 'takes the place' of v_i.

  More precisely, whenever there was an edge joining v to $w \in V(G)$ there is an edge joining v_{ij} with w, for all $j = 1, \cdots, t$.

Blowups of graphs I

- Let G be a graph and $t \in \mathbb{N}$. A t-blowup of a vertex $v_i \in V(G)$ is an independent set $V_{v_i} = \{v_{i1}, v_{i2}, \cdots, v_{it}\}$ that 'takes the place' of v_i.

  More precisely, whenever there was an edge joining v to $w \in V(G)$ there is an edge joining v_{ij} with w, for all $j = 1, \cdots, t$.

- The graph obtained by the t-blowup of v will be called $G(tv)$.

Blowups of graphs I

- Let G be a graph and $t \in \mathbb{N}$. A t-blowup of a vertex $v_i \in V(G)$ is an independent set $V_{v_i} = \{v_{i1}, v_{i2}, \cdots, v_{it}\}$ that 'takes the place' of v_i.

  More precisely, whenever there was an edge joining v to $w \in V(G)$ there is an edge joining v_{ij} with w, for all $j = 1, \cdots, t$.

- The graph obtained by the t-blowup of v will be called $G(tv)$.

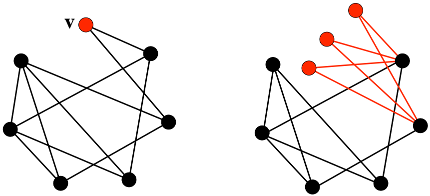


Figure: G and $G(3v)$.

Blowups of graphs II

- In case many blowups are performed to vertices of the same graph then the notation will follow from

$$G(tv)(sw) = G(tv, sw)$$

Blowups of graphs II

- In case many blowups are performed to vertices of the same graph then the notation will follow from

$$G(tv)(sw) = G(tv, sw)$$

- (B & V) Let G be a graph with $V(G) = \{v_1, \cdots, v_n\}$ and $m = wcdim(G, \mathbb{F})$. Let $H = G(t_1v_1, t_2v_2, \cdots, t_nv_n)$, where $t_i \in \mathbb{N}$ for all $i = 1, 2, \cdots, n$.

Blowups of graphs II

- In case many blowups are performed to vertices of the same graph then the notation will follow from

$$G(tv)(sw) = G(tv, sw)$$

- (B & V) Let G be a graph with $V(G) = \{v_1, \cdots, v_n\}$ and $m = wcdim(G, \mathbb{F})$. Let $H = G(t_1 v_1, t_2 v_2, \cdots, t_n v_n)$, where $t_i \in \mathbb{N}$ for all $i = 1, 2, \cdots, n$. Then,

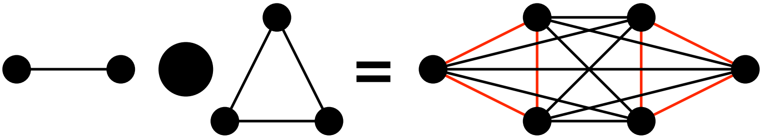
$$wcdim(H, \mathbb{F}) = (m - n) + \sum_{i=1}^{n} t_i$$

Lexicographic products of graphs I

- The lexicographic product of two graphs G and H, denoted $G \bullet H$, is the graph with vertex set $V(G) \times V(H)$ and edges joining (g, h) and (g', h') if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.

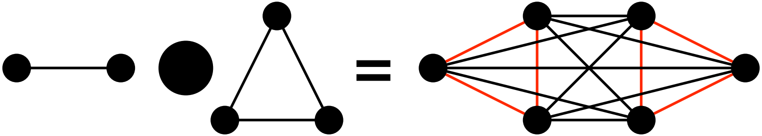
Lexicographic products of graphs I

- The lexicographic product of two graphs G and H, denoted $G \bullet H$, is the graph with vertex set $V(G) \times V(H)$ and edges joining (g, h) and (g', h') if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.



Lexicographic products of graphs I

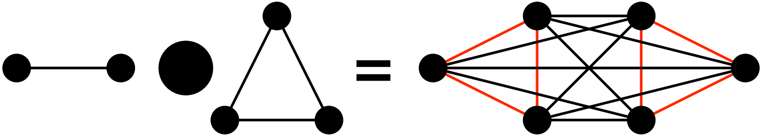
- The lexicographic product of two graphs G and H, denoted $G \bullet H$, is the graph with vertex set $V(G) \times V(H)$ and edges joining (g, h) and (g', h') if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.



- If G is a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $wcdim(G, \mathbb{F}) = m$. Then, using that $G \bullet \overline{K_t} = G(tv_1, tv_2, \cdots, tv_n)$ we get

Lexicographic products of graphs I

- The lexicographic product of two graphs G and H, denoted $G \bullet H$, is the graph with vertex set $V(G) \times V(H)$ and edges joining (g, h) and (g', h') if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.



- If G is a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $wcdim(G, \mathbb{F}) = m$. Then, using that $G \bullet \overline{K_t} = G(tv_1, tv_2, \cdots, tv_n)$ we get

$$wcdim\left(G \bullet \overline{K_t}, \mathbb{F}\right) = m + n(t-1)$$

Lexicographic products of graphs II

- (B & V) Let G and H be graphs with $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \mathbb{F}) = n$, and $wcdim(H, \mathbb{F}) = m$. Then,

Lexicographic products of graphs II

- (B & V) Let G and H be graphs with $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \mathbb{F}) = n$, and $wcdim(H, \mathbb{F}) = m$. Then,

  $$wcdim(G \bullet H, \mathbb{F}) = nb + am - nm + \delta_{m-b+1,i}(n-a) + \delta_{n-a+1,j}(m-b)$$

Lexicographic products of graphs II

- (B & V) Let G and H be graphs with $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \mathbb{F}) = n$, and $wcdim(H, \mathbb{F}) = m$. Then,

  $$wcdim(G \bullet H, \mathbb{F}) = nb + am - nm + \delta_{m-b+1,i}(n-a) + \delta_{n-a+1,j}(m-b)$$

  where δ_{xy} represents the Kronecker delta, and i, j are the number of maximal independent sets of H and G respectively.

Lexicographic products of graphs II

- (B & V) Let G and H be graphs with $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \mathbb{F}) = n$, and $wcdim(H, \mathbb{F}) = m$. Then,

  $$wcdim(G \bullet H, \mathbb{F}) = nb + am - nm + \delta_{m-b+1,i}(n-a) + \delta_{n-a+1,j}(m-b)$$

  where δ_{xy} represents the Kronecker delta, and i, j are the number of maximal independent sets of H and G respectively.

- Let G and H be graphs with more maximal independent sets than vertices, and such that $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \mathbb{F}) = n$ and $wcdim(H, \mathbb{F}) = m$. Then,

  $$wcdim(G \bullet H, \mathbb{F}) = nb + am - nm$$

Generalized quadrangles I

A finite generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying

Generalized quadrangles I

A finite generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying

(i) Each point is incident with $t + 1$ lines $(t \geq 1)$ and two distinct points are incident with at most one line.

Generalized quadrangles I

A finite generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying

(i) Each point is incident with $t + 1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $s + 1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

Generalized quadrangles I

A finite generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying

(i) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \; \mathcal{I} \; M \; \mathcal{I} \; y \; \mathcal{I} \; L$.

Generalized quadrangles I

A finite generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying

(i) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

The integers s and t are called the parameters of the GQ. Also, $\mathcal{S}$ is said to have order (s, t).

Generalized quadrangles II

- Let $\mathcal{S}$ be a generalized quadrangle of order (s, t). Let $G(\mathcal{S})$ be the graph with vertices the points of $\mathcal{S}$ and edges joining points that are collinear in $\mathcal{S}$.

Generalized quadrangles II

- Let $\mathcal{S}$ be a generalized quadrangle of order (s, t). Let $G(\mathcal{S})$ be the graph with vertices the points of $\mathcal{S}$ and edges joining points that are collinear in $\mathcal{S}$.

- (B & V) $\mathit{wcdim}(G(\mathcal{S}), \mathbb{F}) = 2s + 1$ for all fields $\mathbb{F}$.